

# HIGH-ORDER ALGORITHMS FOR RIESZ TYPE TURBULENT DIFFUSION EQUATION

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**ABSTRACT.** Numerical methods for fractional calculus attract increasing interests due to its wide applications in various fields such as physics, mechanics, etc. In this paper, we focus on constructing high-order algorithms for Riesz derivatives, where the convergence orders cover from the second order to the sixth order. Then we apply the established schemes to the Riesz type turbulent diffusion equation (or, Riesz space fractional turbulent diffusion equation). Numerical experiments are displayed which support the theoretical analysis.

## 1. INTRODUCTION

In recent years, fractional calculus has attracted increasing interests due to its potential applications in physics, mechanics, etc. For more details, see the review articles [1] and the recent studies [2, 3, 4, 5, 6, 7, 8], and references cited therein. As far as we know, the Riemann-Liouville (RL) derivative and Caputo derivative are commonly used, which are respectively defined below

$${}_{RL}D_{a,x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\alpha-1} f(\xi) d\xi, \quad x \in (a,b), \quad n-1 < \alpha < n \in \mathbb{Z}^+,$$

and,

$${}_CD_{a,x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \quad x \in (a,b), \quad n-1 < \alpha < n \in \mathbb{Z}^+.$$

These two definitions are generally not equivalent.

A linear combination of the left RL derivative (defined above) and the right RL derivative defined below

$${}_{RL}D_{x,b}^{\alpha}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (\xi-x)^{n-\alpha-1} f(\xi) d\xi, \quad x \in (a,b), \quad n-1 < \alpha < n \in \mathbb{Z}^+,$$

defines a new derivative, i.e., Riesz derivative,

$$\frac{d^{\alpha}f(x)}{d|x|^{\alpha}} = -\frac{1}{2\cos(\pi\alpha/2)} ({}_{RL}D_{a,x}^{\alpha} + {}_{RL}D_{x,b}^{\alpha}) f(x).$$

The left RL derivative reflects the memory (or, the dependence on the history), while the right RL derivative “looks forward to” the future (or, the dependence

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upon the future). So the Riesz derivative value of  $f(x)$  at  $x$  relies on the whole space  $(a, b)$ , but the values are different at different  $x \in (a, b)$ . From an angle of application, the case with  $\alpha \in (0, 2)$  is mostly often attracted attention. If  $\alpha = 0$ , then  $\frac{d^\alpha f(x)}{d|x|^\alpha} = f(x)$ ; if  $\alpha = 1$ , generally set  $\frac{d^\alpha f(x)}{d|x|^\alpha} = f'(x)$  for convenience; if  $\alpha = 2$ , then it is reduced to  $f''(x)$ .

From the studies available, the high-order algorithms for RL derivatives were established in [9]. In [10], Ding et al. established the sixth-order, eighth-order, tenth-order and twelfth-order schemes for the Riesz derivative by using the Fourier analysis. Then they used the former two schemes to the Riesz space fractional reaction-dispersion equation where the rigorous error analyses were given. The other odd-order (third-order, fifth-order, seventh-order, ninth-order, eleventh-order) schemes can be established from [10] by choosing the different parameters therein. In spite of these, it is absolutely necessary to construct intuitionist and straightforward schemes for the Riesz derivatives. Here we find an interesting and enlightening way to establish high-order schemes (from 2nd-order to 6th-order) by using the corresponding generating functions [9]. Then we use these schemes to solve the Riesz space fractional turbulent diffusion equation.

In the following, we briefly introduce fractional modelling in this respect. To the best of our knowledge, from the known first Fick's law

$$J(x, t) = d_1 u(x, t) - d_2 \frac{\partial u(x, t)}{\partial x}, \quad d_1 > 0, \quad d_2 > 0,$$

one can obtain the following advection-diffusion equation,

$$\frac{\partial u(x, t)}{\partial t} = -d_1 \frac{\partial u(x, t)}{\partial x} + d_2 \frac{\partial^2 u(x, t)}{\partial x^2}.$$

If the advection-diffusion process at any position  $x \in (a, b)$  relies on the whole space  $(a, b)$  (i.e., long-range interactions), then the classical Fick's law does not work well yet. However, the fractional derivative can well characterize such long-range interactions. Now we generalize the typical Fick's law to a fractional version,

$$J_\alpha(x, t) = d_1 u(x, t) - d_2 \frac{\partial u(x, t)}{\partial x} - d_\alpha \frac{\partial^{\alpha-1} u(x, t)}{\partial |x|^{\alpha-1}}, \quad \alpha \in (0, 1),$$

where  $d_1, d_2, d_\alpha$  are positive constants. It follows that the Riesz space fractional turbulent diffusion equation (or, Riesz type turbulent diffusion equation) is obtained,

$$\frac{\partial u(x, t)}{\partial t} = -d_1 \frac{\partial u(x, t)}{\partial x} + d_2 \frac{\partial^2 u(x, t)}{\partial x^2} + d_\alpha \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha}.$$

If there is a source term, then one has

$$(1.1) \quad \frac{\partial u(x, t)}{\partial t} = -d_1 \frac{\partial u(x, t)}{\partial x} + d_2 \frac{\partial^2 u(x, t)}{\partial x^2} + d_\alpha \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + s(x, t),$$

where the Riesz partial derivative with order  $\alpha \in (0, 1)$  is given by

$$(1.2) \quad \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = -\frac{1}{2 \cos(\pi\alpha/2)} ({}_{RL}D_{a,x}^\alpha + {}_{RL}D_{x,b}^\alpha) u(x, t), \quad 0 < \alpha < 1.$$

Equ. (1.1) is subject to the following initial value condition,

$$u(x, 0) = \varphi(x), \quad x \in [a, b],$$

and the boundary value conditions (here choosing the homogeneous condition for brevity),

$$u(a, t) = 0, \quad u(b, t) = 0, \quad t \geq 0.$$

The remainder of this paper is outlined as follows. In Section 2, five kinds of high-order (2nd-order, 3rd-order,  $\dots$ , 6th-order) algorithms for the Riesz derivatives are developed. In the next section, we apply the derived schemes to the Riesz-type turbulent diffusion equation (1.1). Here we only use the 2nd-order, 4th-order, 6th-order schemes to (1.1). The convergence orders are  $\mathcal{O}(\tau^2 + h^2)$ ,  $\mathcal{O}(\tau^2 + h^4)$ , and  $\mathcal{O}(\tau^2 + h^6)$ , where  $\tau$  and  $h$  are temporal and spatial stepsizes, respectively. In Section 4, numerical examples are displayed which support the theoretical analysis. The last section concludes this article.

## 2. HIGH-ORDER NUMERICAL SCHEMES

If  $f^{(k)}(a+) = 0$  ( $k = 0, 1, \dots, p-1$ ), then it follows from [9] that the left RL derivative has the following approximations

$$(2.1) \quad {}_{RL}D_{a,x}^\alpha f(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} f(x - \ell h) + \mathcal{O}(h^p),$$

in which  $h$  is the steplength. Here we only show interests in  $p = 2, 3, 4, 5, 6$ .

The convolution (or weight) coefficients  $\varpi_{p,\ell}^{(\alpha)}$  in the above equations are those of the Taylor series expansions of the corresponding generating functions  $W_p^{(\alpha)}(z)$ ,

$$W_p^{(\alpha)}(z) = \sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} z^\ell, \quad \alpha \in (0, 2),$$

where

$$\begin{aligned} W_2^{(\alpha)}(z) &= \left( \frac{3}{2} - 2z + \frac{1}{2}z^2 \right)^\alpha, \\ W_3^{(\alpha)}(z) &= \left( \frac{11}{6} - 3z + \frac{3}{2}z^2 - \frac{1}{3}z^3 \right)^\alpha, \\ W_4^{(\alpha)}(z) &= \left( \frac{25}{12} - 4z + 3z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4 \right)^\alpha, \\ W_5^{(\alpha)}(z) &= \left( \frac{137}{60} - 5z + 5z^2 - \frac{10}{3}z^3 + \frac{5}{4}z^4 - \frac{1}{5}z^5 \right)^\alpha, \\ W_6^{(\alpha)}(z) &= \left( \frac{147}{60} - 6z + \frac{15}{2}z^2 - \frac{20}{3}z^3 + \frac{15}{4}z^4 - \frac{6}{5}z^5 + \frac{1}{6}z^6 \right)^\alpha. \end{aligned}$$

By tedious but direct calculations, one has

$$\begin{aligned} \varpi_{2,\ell}^{(\alpha)} &= \left( \frac{3}{2} \right)^\alpha \sum_{\ell_1=0}^{\ell} \left( \frac{1}{3} \right)^{\ell_1} \varpi_{1,\ell_1}^{(\alpha)} \varpi_{1,\ell-\ell_1}^{(\alpha)}, \\ \varpi_{3,\ell}^{(\alpha)} &= \left( \frac{11}{6} \right)^\alpha \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\lfloor \frac{1}{2}\ell_1 \rfloor} (-1)^{\ell_2} \left( \frac{7}{11} \right)^{\ell_1-\ell_2} \left( \frac{2}{7} \right)^{\ell_2} \frac{(\ell_1-\ell_2)!}{\ell_2!(\ell_1-2\ell_2)!} \varpi_{1,\ell-\ell_1}^{(\alpha)} \varpi_{1,\ell_1-\ell_2}^{(\alpha)}, \end{aligned}$$

$$\begin{aligned}
\varpi_{4,\ell}^{(\alpha)} &= \left(\frac{25}{12}\right)^\alpha \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\lfloor \frac{2}{3}\ell_1 \rfloor} \sum_{\ell_3=\max\{0, 2\ell_2-\ell_1\}}^{\lfloor \frac{1}{2}\ell_2 \rfloor} (-1)^{\ell_2} \left(\frac{23}{25}\right)^{\ell_1-\ell_2} \left(\frac{13}{23}\right)^{\ell_2-\ell_3} \left(\frac{3}{13}\right)^{\ell_3} \times \\
&\quad \frac{(\ell_1 - \ell_2)!}{\ell_3! (\ell_2 - 2\ell_3)! (\ell_1 + \ell_3 - 2\ell_2)!} \varpi_{1,\ell-\ell_1}^{(\alpha)} \varpi_{1,\ell_1-\ell_2}^{(\alpha)}, \\
\varpi_{5,\ell}^{(\alpha)} &= \left(\frac{137}{60}\right)^\alpha \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\lfloor \frac{3}{4}\ell_1 \rfloor} \sum_{\ell_3=\max\{0, 2\ell_2-\ell_1\}}^{\lfloor \frac{2}{3}\ell_2 \rfloor} \sum_{\ell_4=\max\{0, 2\ell_3-\ell_2\}}^{\lfloor \frac{1}{2}\ell_3 \rfloor} (-1)^{\ell_2} \left(\frac{163}{137}\right)^{\ell_1-\ell_2} \left(\frac{137}{163}\right)^{\ell_2-\ell_3} \\
&\quad \times \left(\frac{63}{137}\right)^{\ell_3-\ell_4} \left(\frac{4}{21}\right)^{\ell_4} \frac{(\ell_1 - \ell_2)!}{\ell_4! (\ell_3 - 2\ell_4)! (\ell_1 + \ell_3 - 2\ell_2)! (\ell_2 + \ell_4 - 2\ell_3)!} \varpi_{1,\ell-\ell_1}^{(\alpha)} \varpi_{1,\ell_1-\ell_2}^{(\alpha)},
\end{aligned}$$

and

$$\begin{aligned}
\varpi_{6,\ell}^{(\alpha)} &= \left(\frac{147}{60}\right)^\alpha \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\lfloor \frac{4}{5}\ell_1 \rfloor} \sum_{\ell_3=\max\{0, 2\ell_2-\ell_1\}}^{\lfloor \frac{3}{4}\ell_2 \rfloor} \sum_{\ell_4=\max\{0, 2\ell_3-\ell_2\}}^{\lfloor \frac{2}{3}\ell_3 \rfloor} \sum_{\ell_5=\max\{0, 2\ell_4-\ell_3\}}^{\lfloor \frac{1}{2}\ell_4 \rfloor} \\
&\quad (-1)^{\ell_2} \left(\frac{213}{147}\right)^{\ell_1-\ell_2} \left(\frac{237}{213}\right)^{\ell_2-\ell_3} \left(\frac{163}{237}\right)^{\ell_3-\ell_4} \left(\frac{62}{163}\right)^{\ell_4-\ell_5} \left(\frac{5}{31}\right)^{\ell_5} \times \\
&\quad \frac{(\ell_1 - \ell_2)!}{\ell_5! (\ell_4 - 2\ell_5)! (\ell_1 + \ell_3 - 2\ell_2)! (\ell_2 + \ell_4 - 2\ell_3)! (\ell_3 + \ell_5 - 2\ell_4)!} \varpi_{1,\ell-\ell_1}^{(\alpha)} \varpi_{1,\ell_1-\ell_2}^{(\alpha)},
\end{aligned}$$

$$\ell = 0, 1, \dots$$

Here  $\varpi_{1,j}^{(\alpha)}$  is the first order coefficients defined by  $\varpi_{1,j}^{(\alpha)} = (-1)^j \frac{\Gamma(1+\alpha)}{\Gamma(j+1)\Gamma(1+\alpha-j)}$ ,  $j = 0, 1, \dots$ . If  $j \geq 2$ , then  $\varpi_{1,j-1}^{(\alpha)} \leq \varpi_{1,j}^{(\alpha)}$  for  $\alpha \in (0, 1)$  whilst  $\varpi_{1,j-1}^{(\alpha)} \geq \varpi_{1,j}^{(\alpha)}$  for  $\alpha \in (1, 2)$ . See [11] for more information.

On the other hand, if  $f^{(k)}(b-) = 0$  ( $k = 0, 1, \dots, p-1$ ), then one has the approximations below,

$$(2.2) \quad {}_{RL}D_{x,b}^\alpha f(x) = \frac{1}{h^\alpha} \sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} f(x + \ell h) + \mathcal{O}(h^p), \quad p = 2, \dots, 6,$$

where  $h$  is also the stepsize.

Based on (2.1) and (2.2), if  $f(x)$ , together with its derivatives, has homogeneous boundary value conditions, one easily gets

$$(2.3) \quad \frac{\partial^\alpha f(x)}{\partial |x|^\alpha} = -\frac{1}{2 \cos(\pi\alpha/2) h^\alpha} \sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} (f(x - \ell h) + f(x + \ell h)) + \mathcal{O}(h^p).$$

Since  $\alpha \in (0, 1)$  is commonly used, we limit our interests in  $\alpha \in (0, 1)$ . The case  $\alpha \in (1, 2)$  can be similarly studied. When  $\alpha = 1$ , we often set  $\frac{\partial^\alpha f(x)}{\partial |x|^\alpha} = f'(x)$  which is the trivial case so is omitted here.

The properties of the convolution coefficients  $\varpi_{p,\ell}^{(\alpha)}$  are very important for constructing effective numerical algorithms for Riemann-Liouville time fractional differential equations and Riemann-Liouville (or Riesz) space fractional differential equations. So we first study these convolution coefficients. For the space fractional differential equations, we use the following properties of the coefficients  $\varpi_{p,\ell}^{(\alpha)}$  to

show the stability and convergence of the derived algorithms. Here we focus on studying the case  $\alpha \in (0, 1)$ .

**Theorem 2.1.** *For  $0 < \alpha < 1$ , then the following inequalities hold:*

$$\sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} \cos(\ell\theta) \geq 0, \quad \theta \in [-\pi, \pi], \quad p = 2, 3, 5, 6.$$

*Proof.* We only prove  $p = 2$ , the other cases can be almost similarly shown so are left out here. Let

$$f_1(\alpha, \theta) = \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\alpha)} \cos(\ell\theta),$$

which can be expanded as

$$\begin{aligned} f_1(\alpha, \theta) &= \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\alpha)} \cos(\ell\theta) = \frac{1}{2} \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\alpha)} (\exp(i\ell\theta) + \exp(-i\ell\theta)) \\ &= \frac{1}{2} \left[ (1 - \exp(i\theta))^\alpha \left( \frac{3}{2} - \frac{1}{2} \exp(i\theta) \right)^\alpha + (1 - \exp(-i\theta))^\alpha \left( \frac{3}{2} - \frac{1}{2} \exp(-i\theta) \right)^\alpha \right]. \end{aligned}$$

Note that  $f_1(\alpha, \theta)$  is a real-value and even function, so we need only consider  $\theta \in [0, \pi]$ .

Using the following equations

$$(1 - \exp(\pm i\theta))^\alpha = \left( 2 \sin \frac{\theta}{2} \right)^\alpha \exp \left( \pm i\alpha \left( \frac{\theta - \pi}{2} \right) \right)$$

and

$$(x - yi)^\alpha = (x^2 + y^2)^{\frac{\alpha}{2}} \exp(i\alpha\phi), \quad \phi = -\arctan \frac{y}{x},$$

we can rewrite  $f_1(\alpha, \theta)$  as

$$f_1(\alpha, \theta) = \left( 2 \sin \frac{\theta}{2} \right)^\alpha (\lambda_1^2(\theta) + \mu_1^2(\theta))^{\frac{\alpha}{2}} \cos \alpha \left( \frac{\theta - \pi}{2} + \phi_1 \right),$$

where

$$\lambda_1(\theta) = 3 - \cos \theta, \quad \mu_1(\theta) = \sin \theta, \quad \phi_1 = -\arctan \frac{\mu_1(\theta)}{\lambda_1(\theta)}.$$

Let

$$z(\theta) = \frac{\theta - \pi}{2} + \phi_1, \quad 0 \leq \theta \leq \pi.$$

Then

$$z'(\theta) = \left( \frac{\theta - \pi}{2} + \phi_1 \right)' = \frac{3 \sin^2 \left( \frac{\theta}{2} \right)}{1 + 3 \sin^2 \left( \frac{\theta}{2} \right)} \geq 0.$$

Hence  $z(\theta)$  is an increasing function in  $[0, \pi]$  and

$$z_{\min}(\theta) = z(0) = -\frac{\pi}{2}, \quad z_{\max}(\theta) = z(\pi) = 0.$$

So  $\alpha \in (0, 1)$  and  $\theta \in [0, \pi]$  imply  $\cos \alpha \left( \frac{\theta - \pi}{2} + \phi_1 \right) \geq 0$ . Furthermore one has

$$f_1(\alpha, \theta) = \left( 2 \sin \frac{\theta}{2} \right)^\alpha (\lambda_1^2(\theta) + \mu_1^2(\theta))^{\frac{\alpha}{2}} \cos \alpha \left( \frac{\theta - \pi}{2} + \phi_1 \right) \geq 0.$$

All this ends the proof.  $\square$

For  $p = 4$ ,  $\alpha$  can not attain to 1. But we have following theorem.

**Theorem 2.2.** *If  $0 < \alpha \leq \frac{\pi}{\pi - \arccos \frac{1}{5} + 2 \arctan \frac{191\sqrt{6}}{317}} \approx 0.8439$ , then the following inequality holds:*

$$\sum_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} \cos(\ell\theta) \geq 0, \quad \theta \in [-\pi, \pi].$$

*Proof.* Let  $f_2(\alpha, \theta) = \sum_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} \cos(\ell\theta)$ . By almost the same reasoning as that of Theorem 2.1, we can get

$$f_2(\alpha, \theta) = \left(2 \sin \frac{\theta}{2}\right)^{\alpha} (\lambda_2^2(\theta) + \mu_2^2(\theta))^{\frac{\alpha}{2}} \cos \alpha \left(\frac{\theta - \pi}{2} + \phi_2\right),$$

where

$$\begin{aligned} \lambda_2(\theta) &= 25 - 23 \cos \theta + 13 \cos 2\theta - 3 \cos 3\theta, \\ \mu_2(\theta) &= 23 \sin \theta - 13 \sin 2\theta + 3 \sin 3\theta, \quad \phi_2 = -\arctan \frac{\mu_2(\theta)}{\lambda_2(\theta)}. \end{aligned}$$

Since

$$\lambda_2(\theta) = 14 \left(\cos(\theta) - \frac{1}{2}\right)^2 + 24 \cos^2(\theta) \sin^2\left(\frac{\theta}{2}\right) + \frac{17}{2} > 0,$$

and

$$\mu_2(\theta) = \left[12 \left(\cos(\theta) - \frac{13}{12}\right)^2 + \frac{71}{24}\right] \sin(\theta) \geq 0,$$

so  $\phi_2 \in [-\frac{\pi}{2}, 0]$ . We need only consider  $0 \leq \theta \leq \pi$ , therefore  $-\pi \leq \alpha \left(\frac{\theta - \pi}{2} + \phi_2\right) \leq 0$ .

Obviously, if  $\cos \alpha \left(\frac{\theta - \pi}{2} + \phi_2\right) \geq 0$ , then  $f_2(\alpha, \theta) \geq 0$ . A sufficient condition for  $\cos \alpha \left(\frac{\theta - \pi}{2} + \phi_2\right) \geq 0$  is

$$-\frac{\pi}{2} \leq \min_{\theta \in [0, \pi]} \alpha \left(\frac{\theta - \pi}{2} + \phi_2\right) \leq 0,$$

i.e.,

$$0 < \alpha \leq \min_{\theta \in [0, \pi]} \left\{ \frac{\pi}{\pi - \theta - 2\phi_2} \right\}.$$

Let  $y(\theta) = \pi - \theta - 2\phi_2$ , then

$$y'(\theta) = \frac{1920(5 \cos \theta - 1) \sin^4\left(\frac{\theta}{2}\right)}{a_2^2(\theta) + b_2^2(\theta)}.$$

It is clear that  $\theta = \arccos \frac{1}{5}$  is a unique maximum point of  $y(\theta)$  when  $\theta \in [0, \pi]$ , i.e.,

$$y_{\max}(\theta) = y_{\max}\left(\arccos \frac{1}{5}\right) = \pi - \arccos \frac{1}{5} + 2 \arctan \frac{191\sqrt{6}}{317},$$

it follows that

$$\min_{\theta \in [0, \pi]} \left\{ \frac{\pi}{\pi - \theta - 2\phi_2} \right\} = \frac{\pi}{\pi - \arccos \frac{1}{5} + 2 \arctan \frac{191\sqrt{6}}{317}} \approx 0.8439,$$

i.e.,

$$0 < \alpha \leq 0.8439.$$

This finishes the proof.  $\square$

Theorems 2.1 and 2.2 are very suitable for numerically analyzing Riemann-Liouville space fractional partial differential equations. But for numerically analyzing Riemann-Liouville time fractional partial differential equations, the monotonicity of the coefficients  $\varpi_{p,\ell}^{(\alpha)}$  ( $p = 1, 2, 3, 4, 5, 6$ ) is often necessary. As far as we know, only the monotonicity of the coefficients  $\varpi_{1,\ell}^{(\alpha)}$  for  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ ,  $\varpi_{2,\ell}^{(\alpha)}$  for  $\alpha \in (0, 1)$  are available [11]. In the following, we study the rest cases.

**Theorem 2.3.** *The second-order coefficients  $\varpi_{2,\ell}^{(\alpha)}$  ( $\ell = 0, 1, \dots$ ) satisfy*

$$\begin{aligned} (1) \quad & \varpi_{2,0}^{(\alpha)} = \left(\frac{3}{2}\right)^\alpha > 0, \quad \varpi_{2,1}^{(\alpha)} = -\frac{4\alpha}{3} \left(\frac{3}{2}\right)^\alpha < 0, \\ & \varpi_{2,2}^{(\alpha)} = \frac{\alpha(8\alpha-3)}{9} \left(\frac{3}{2}\right)^\alpha, \quad \varpi_{2,3}^{(\alpha)} = -\frac{4\alpha(\alpha-1)(8\alpha-7)}{81} \left(\frac{3}{2}\right)^\alpha, \\ & \varpi_{2,4}^{(\alpha)} = \frac{\alpha(\alpha-1)(64\alpha^2-176\alpha+123)}{486} \left(\frac{3}{2}\right)^\alpha, \\ & \vdots \end{aligned}$$

(2) When  $0 < \alpha < 1$ ,  $\varpi_{2,\ell}^{(\alpha)} < 0$  and  $\varpi_{2,\ell}^{(\alpha)} < \varpi_{2,\ell+1}^{(\alpha)}$  hold for  $\ell \geq 4$ ,

(3) When  $1 < \alpha < 2$ ,  $\varpi_{2,\ell}^{(\alpha)} > 0$  and  $\varpi_{2,\ell}^{(\alpha)} > \varpi_{2,\ell+1}^{(\alpha)}$  hold for  $\ell \geq 5$ .

*Proof.* See the Appendix A. □

Although it is not facile to prove the monotonicity of the coefficients  $\varpi_{p,\ell}^{(\alpha)}$ ,  $p = 3, \dots, 6$ , we can explicitly write their expressions, see the beginning part of section 2 for more details. These coefficients are explicitly expressed which are beneficial for numerical calculations. Besides, through a huge of numerical simulations, one can find the monotonicity of the coefficients  $\varpi_{p,\ell}^{(\alpha)}$ ,  $p = 3, \dots, 5$ .

Here, we only list very limited figures to show the monotonicity of the coefficients for  $p = 3, 4, 5$ . Figs. 1 and 2 show the monotonicity of the coefficients  $\varpi_{3,\ell}^{(\alpha)}$  for  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ . Figs. 3 and 4 display the monotonicity of the coefficients  $\varpi_{4,\ell}^{(\alpha)}$  for  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ . Figs. 5 and 6 present the monotonicity of the coefficients  $\varpi_{5,\ell}^{(\alpha)}$  for  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ . But through the numerical simulations,  $\varpi_{6,\ell}^{(\alpha)}$ ,  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$  seem not to have the monotonicity.

In the following, we give the conjecture which is seemingly primary but hard to prove.

### Conjecture

- (1) If  $0 < \alpha < 1$ , then  $\varpi_{3,\ell}^{(\alpha)} \leq \varpi_{3,\ell+1}^{(\alpha)}$  holds for  $\ell \geq 4$ ,  $\varpi_{4,\ell}^{(\alpha)} \leq \varpi_{4,\ell+1}^{(\alpha)}$  holds for  $\ell \geq 7$ , and  $\varpi_{5,\ell}^{(\alpha)} \leq \varpi_{5,\ell+1}^{(\alpha)}$  for  $\ell \geq 12$ .
- (2) If  $1 < \alpha < 2$ , then  $\varpi_{3,\ell}^{(\alpha)} \geq \varpi_{3,\ell+1}^{(\alpha)}$  holds for  $\ell \geq 7$ ,  $\varpi_{4,\ell}^{(\alpha)} \geq \varpi_{4,\ell+1}^{(\alpha)}$  holds for  $\ell \geq 12$ , and  $\varpi_{5,\ell}^{(\alpha)} \geq \varpi_{5,\ell+1}^{(\alpha)}$  for  $\ell \geq 16$ .

Besides their monotonicity, studying bounds of these coefficients is also of importance, which can be used to analyze the stability and convergence for time

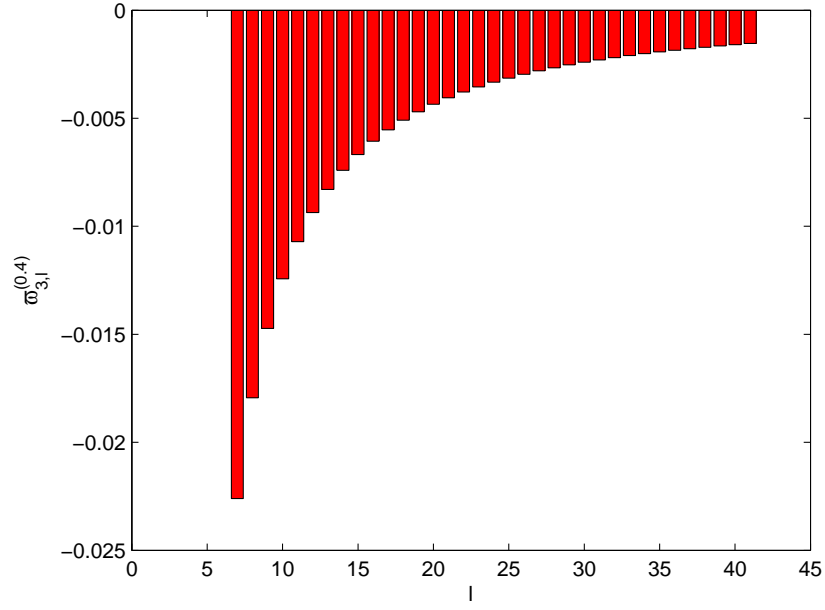


FIGURE 1. The values of coefficient  $\varpi_{3,\ell}^{(\alpha)}$  ( $\ell = 4, 5, \dots$ ) for  $\alpha = 0.4$ .

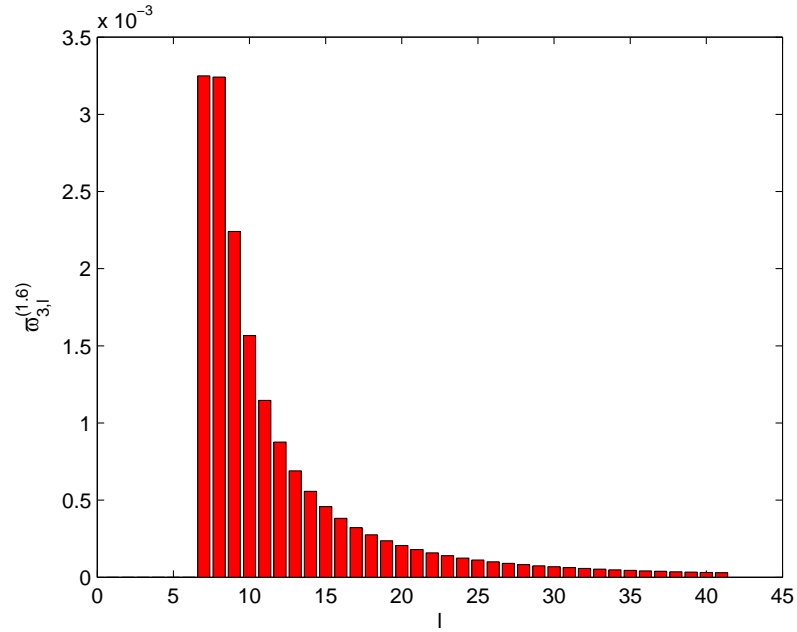


FIGURE 2. The values of coefficient  $\varpi_{3,\ell}^{(\alpha)}$  ( $\ell = 7, 8, \dots$ ) for  $\alpha = 1.6$ .



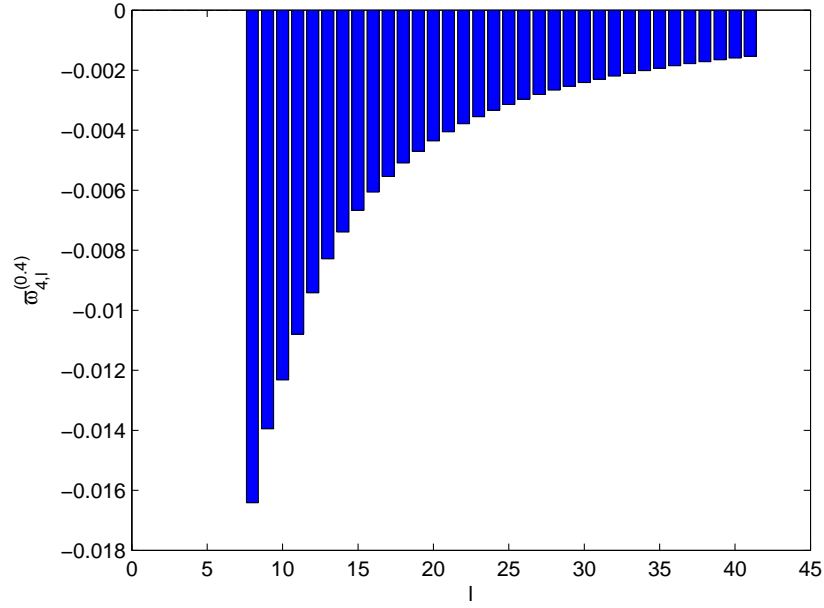


FIGURE 3. The values of coefficient  $\varpi_{4,\ell}^{(\alpha)}$  ( $\ell = 7, 8, \dots$ ) for  $\alpha = 0.4$ .

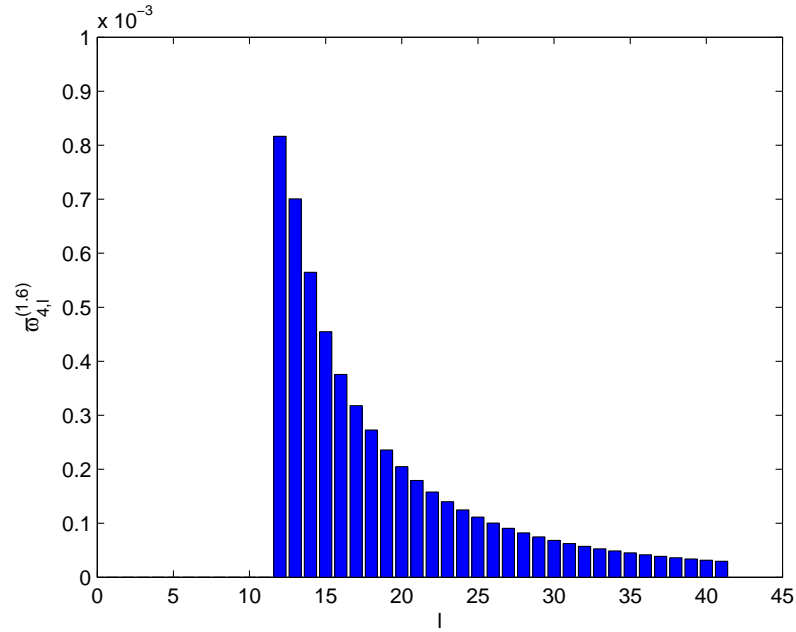


FIGURE 4. The values of coefficient  $\varpi_{4,\ell}^{(\alpha)}$  ( $\ell = 10, 11, \dots$ ) for  $\alpha = 1.6$ .

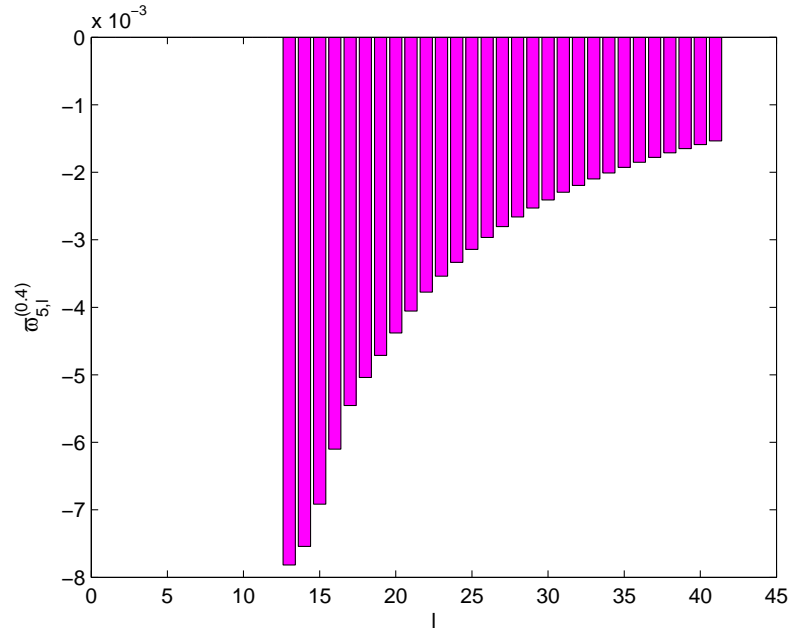


FIGURE 5. The values of coefficient  $\varpi_{5,\ell}^{(\alpha)}$  ( $\ell = 12, 13, \dots$ ) for  $\alpha = 0.4$ .

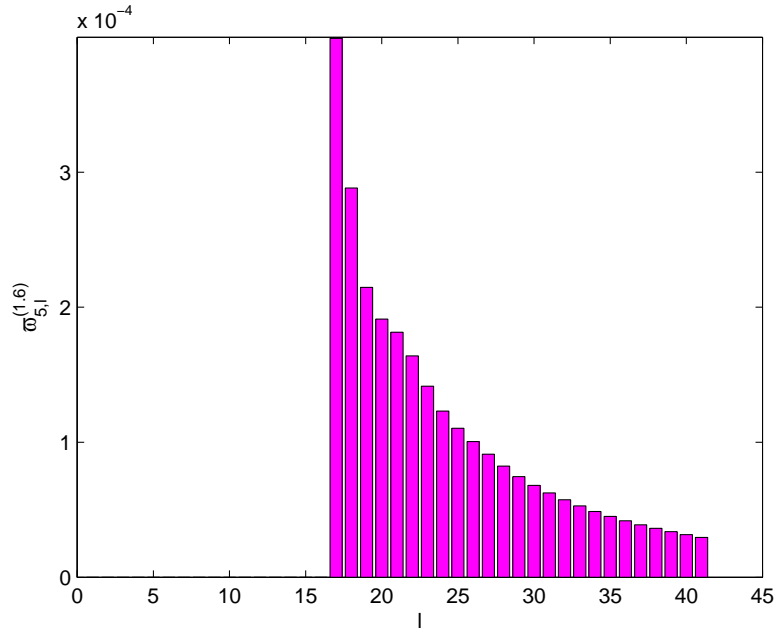


FIGURE 6. The values of coefficient  $\varpi_{5,\ell}^{(\alpha)}$  ( $\ell = 16, 17, \dots$ ) for  $\alpha = 1.6$ .

fractional differential equations. In [12], Dimitrov given the bounds for first-order  $\varpi_{1,\ell}^{(\alpha)}$ , ( $0 < \alpha < 1$ ) below:

**Theorem 2.4.** *The first-order coefficients  $\varpi_{1,\ell}^{(\alpha)}$ , ( $0 < \alpha < 1$ ) satisfy*

$$(1) \quad \tilde{B}_1^L(\alpha, \ell) < \left| \varpi_{1,\ell}^{(\alpha)} \right| < B_1^R(\alpha, \ell), \quad \ell \geq 3,$$

$$\text{where } \tilde{B}_1^L(\alpha, \ell) = \exp \left( -(\alpha+1)^2 \left( \frac{\pi^2}{6} - \frac{5}{4} \right) \right) \frac{\alpha(1-\alpha)2^\alpha}{\ell^{\alpha+1}}, \quad B_1^R(\alpha, \ell) = \frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}},$$

$$(2) \quad \tilde{S}_1^L(\alpha, \ell) < \sum_{k=\ell}^{\infty} \left| \varpi_{1,k}^{(\alpha)} \right| < S_1^R(\alpha, \ell), \quad \ell \geq 3,$$

$$\text{where } \tilde{S}_1^L(\alpha, \ell) = \frac{1-\alpha}{5} \left( \frac{2}{\ell} \right)^\alpha, \quad S_1^R(\alpha, \ell) = 2 \left( \frac{2}{\ell} \right)^\alpha.$$

In this paper, we can give tighter estimates for the lower bounds. See the following theorem.

**Theorem 2.5.** *The first-order coefficients  $\varpi_{1,\ell}^{(\alpha)}$ , ( $0 < \alpha < 1$ ) satisfy*

$$(1) \quad B_1^L(\alpha, \ell) < \left| \varpi_{1,\ell}^{(\alpha)} \right| < B_1^R(\alpha, \ell), \quad \ell \geq 3, \text{ where } B_1^L(\alpha, \ell) = \frac{\alpha(1-\alpha)}{2} \left( \frac{2}{\ell} \right)^{2(\alpha+1)},$$

$$(2) \quad S_1^L(\alpha, \ell) < \sum_{k=\ell}^{\infty} \left| \varpi_{1,k}^{(\alpha)} \right| < S_1^R(\alpha, \ell), \quad \ell \geq 3, \text{ where } S_1^L(\alpha, \ell) = \frac{\alpha(1-\alpha)}{2\alpha+1} \left( \frac{2}{\ell} \right)^{2\alpha+1}.$$

Next, we list the comparison theorem for  $\tilde{B}_1^L(\alpha, \ell)$  and  $B_1^L(\alpha, \ell)$ ,  $\tilde{S}_1^L(\alpha, \ell)$  and  $S_1^L(\alpha, \ell)$ , respectively.

**Theorem 2.6.** *The following inequalities hold:*

$$(1) \quad B_1^L(\alpha, 3) < \tilde{B}_1^L(\alpha, 3) \text{ for } 0 < \alpha < \frac{12 \ln \frac{3}{2}}{2\pi^2 - 15} - 1 \approx 0.0267; \quad B_1^L(\alpha, 4) < \tilde{B}_1^L(\alpha, 4) \text{ for}$$

$$0 < \alpha < \frac{12 \ln 2}{2\pi^2 - 15} - 1 \approx 0.7551; \quad B_1^L(\alpha, \ell) < \tilde{B}_1^L(\alpha, \ell) \text{ for } 0 < \alpha < 1 \text{ and } \ell \geq 5;$$

$$(2) \quad S_1^L(\alpha, \ell) < \tilde{S}_1^L(\alpha, \ell) \text{ for } 0 < \alpha < 1 \text{ and } \ell \geq 3.$$

Finally, we give the bounds for the other coefficients.

**Theorem 2.7.** *The first-order coefficients  $\varpi_{1,\ell}^{(1+\alpha)}$ , ( $0 < \alpha < 1$ ) satisfy*

$$(1) \quad \overline{B}_1^L(1+\alpha, \ell) < \left| \varpi_{1,\ell}^{(1+\alpha)} \right| < \overline{B}_1^R(1+\alpha, \ell), \quad \ell \geq 4,$$

$$\text{where } \overline{B}_1^L(1+\alpha, \ell) = \frac{(1-\alpha)\alpha(1+\alpha)}{6} \left( \frac{3}{\ell} \right)^{2(2+\alpha)}, \quad \overline{B}_1^R(1+\alpha, \ell) = \frac{\alpha(1+\alpha)}{2} \left( \frac{3}{\ell+1} \right)^{2+\alpha},$$

$$(2) \quad \overline{S}_1^L(1+\alpha, \ell) < \sum_{k=\ell}^{\infty} \left| \varpi_{1,k}^{(1+\alpha)} \right| < \overline{S}_1^R(1+\alpha, \ell), \quad \ell \geq 4,$$

$$\text{where } \overline{S}_1^L(1+\alpha, \ell) = \frac{(1-\alpha)\alpha(1+\alpha)}{2(3+2\alpha)} \left( \frac{3}{\ell} \right)^{3+2\alpha}, \quad \overline{S}_1^R(1+\alpha, \ell) = \frac{3\alpha}{2} \left( \frac{3}{\ell} \right)^{1+\alpha}.$$

**Theorem 2.8.** *The second-order coefficients  $\varpi_{2,\ell}^{(\alpha)}$  and  $\varpi_{2,\ell}^{(\alpha+1)}$  for  $0 < \alpha < 1$  satisfy*

$$(1) \quad B_2^L(\alpha, \ell) < \left| \varpi_{2,\ell}^{(\alpha)} \right| < B_2^R(\alpha, \ell), \quad \ell \geq 4,$$

$$\text{where } B_2^L(\alpha, \ell) = \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha(1-\alpha)}{2} \left(\frac{2}{\ell}\right)^{2(1+\alpha)} - \left(1 - \left(\frac{1}{3}\right)^{\ell-1}\right) \frac{\alpha^2 2^{2\alpha+1}}{1 + (\alpha+1)\ell} \right],$$

$$B_2^R(\alpha, \ell) = \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}} - \frac{\alpha^2(1-\alpha)^2 4^{2\alpha+1}}{2} \left(1 - \left(\frac{1}{3}\right)^{\ell-1}\right) \left(\frac{2}{\ell}\right)^{4(\alpha+1)} \right].$$

$$(2) \quad \overline{B}_2^L(1+\alpha, \ell) < \left| \varpi_{2,\ell}^{(1+\alpha)} \right| < \overline{B}_2^R(1+\alpha, \ell), \quad \ell \geq 4,$$

$$\begin{aligned} \text{where } \overline{B}_2^L(1+\alpha, \ell) &= \left(\frac{3}{2}\right)^{1+\alpha} \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{(1-\alpha)\alpha(1+\alpha)}{6} \left(\frac{3}{\ell}\right)^{2(2+\alpha)} \right. \\ &\quad \left. + \frac{(1-\alpha)^2 \alpha^2 (1+\alpha)^2}{216} \left(1 - \left(\frac{1}{3}\right)^{\ell-3}\right) \left(\frac{6}{\ell}\right)^{4(2+\alpha)} - \frac{\alpha(1+\alpha)^2}{2} \left(\frac{1}{3} + \left(\frac{1}{3}\right)^{\ell-1}\right) \left(\frac{3}{\ell}\right)^{2+\alpha} \right], \\ \overline{B}_2^R(1+\alpha, \ell) &= \left(\frac{3}{2}\right)^{1+\alpha} \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha(\alpha+1)3^{\alpha+2}}{2(\ell+1)^{\alpha+2}} + \left(1 - \left(\frac{1}{3}\right)^{\ell-3}\right) \frac{\alpha^2(1+\alpha)^2 3^{2(2+\alpha)}}{24(1+(2+\alpha)\ell)} \right. \\ &\quad \left. - \frac{(1-\alpha)\alpha(1+\alpha)^2}{6} \left(\frac{1}{3} + \left(\frac{1}{3}\right)^{\ell-1}\right) \left(\frac{3}{\ell-1}\right)^{2(2+\alpha)} \right]. \end{aligned}$$

All of the proofs for theorems 2.4-2.8 are given in Appendix B.

### 3. NUMERICAL METHODS FOR THE RIESZ-TYPE TURBULENT DIFFUSION EQUATION

Define  $t_k = k\tau$ ,  $k = 0, 1, \dots, N$ , and  $\tau = \frac{T}{N}$  for a given  $T > 0$ ,  $h = \frac{b-a}{M}$  is the equidistant grid size in space,  $x_j = a + jh$ ,  $j = 0, 1, \dots, M$ .

**3.1. The 2nd-order scheme in space.** Firstly, using the Crank-Nicolson method for the Riesz space fractional turbulent diffusion equation (1.1) in time direction, we can obtain

$$\begin{aligned} (3.1) \quad & \frac{u(x_j, t_{k+1}) - u(x_j, t_k)}{\tau} = \frac{1}{2} \left( d_2 \frac{\partial^2 u(x_j, t_{k+1})}{\partial x^2} - d_1 \frac{\partial u(x_j, t_{k+1})}{\partial x} \right. \\ & \quad \left. + d_\alpha \frac{\partial^\alpha u(x_j, t_{k+1})}{\partial |x|^\alpha} + d_2 \frac{\partial^2 u(x_j, t_k)}{\partial x^2} - d_1 \frac{\partial u(x_j, t_k)}{\partial x} \right. \\ & \quad \left. + d_\alpha \frac{\partial^\alpha u(x_j, t_k)}{\partial |x|^\alpha} \right) + s(x_j, t_{k+\frac{1}{2}}) + \mathcal{O}(\tau^2). \end{aligned}$$

Secondly, for the first- and second-order derivatives, we using the following approximations, respectively

$$(3.2) \quad \frac{\partial u(x_j, t_k)}{\partial x} = \mu_x \delta_x u(x_j, t_k) + \mathcal{O}(h^2),$$

and

$$(3.3) \quad \frac{\partial^2 u(x_j, t_k)}{\partial x^2} = \delta_x^2 u(x_j, t_k) + \mathcal{O}(h^2),$$

where  $\mu_x \delta_x$  and  $\delta_x^2$  are defined by  $\mu_x \delta_x u(x_j, t_k) = \frac{u(x_{j+1}, t_k) - u(x_{j-1}, t_k)}{2h}$ , and  $\delta_x^2 u(x_j, t_k) = \frac{u(x_{j+1}, t_k) - 2u(x_j, t_k) + u(x_{j-1}, t_k))}{h^2}$ .

Next, we choose the second-order formula to approximate Riesz derivative,

$$(3.4) \quad \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = -\frac{1}{2 \cos(\pi\alpha/2) h^\alpha} \left( \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\alpha)} u(x - \ell h, t) + \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\alpha)} u(x + \ell h, t) \right) + \mathcal{O}(h^2).$$

Substituting (3.2), (3.3) and (3.4) into (3.1) and removing the high-order term yield

$$(3.5) \quad \begin{aligned} & \left( \frac{d_2}{h^2} + \frac{d_1}{2h} \right) u_{j-1}^{k+1} - \left( \frac{2}{\tau} + \frac{2d_2}{h^2} \right) u_j^{k+1} + \left( \frac{d_2}{h^2} - \frac{d_1}{2h} \right) u_{j+1}^{k+1} \\ & - \nu \left[ \sum_{\ell=0}^j \varpi_{2,\ell}^{(\alpha)} u_{j-\ell}^{k+1} + \sum_{\ell=0}^{M-j} \varpi_{2,\ell}^{(\alpha)} u_{j+\ell}^{k+1} \right] = - \left( \frac{d_2}{h^2} + \frac{d_1}{2h} \right) u_{j-1}^k \\ & - \left( \frac{2}{\tau} - \frac{2d_2}{h^2} \right) u_j^k - \left( \frac{d_2}{h^2} - \frac{d_1}{2h} \right) u_{j+1}^k + \nu \left[ \sum_{\ell=0}^j \varpi_{2,\ell}^{(\alpha)} u_{j-\ell}^k \right. \\ & \left. + \sum_{\ell=0}^{M-j} \varpi_{2,\ell}^{(\alpha)} u_{j+\ell}^k \right] - 2s_j^{k+\frac{1}{2}}, \quad j = 1, \dots, M-1, \quad k = 0, 1, \dots, N-1, \end{aligned}$$

where  $\nu = \frac{d_\alpha}{2 \cos(\pi\alpha/2) h^\alpha}$ .

Next we discuss the stability and convergence of scheme (3.5).

**Theorem 3.1.** *The numerical scheme (3.5) is unconditionally stable and convergent with order  $\mathcal{O}(\tau^2 + h^2)$ .*

*Proof.* Assume that the solution of equation (1.1) can be zero extended to the whole real line  $R$ . Suppose  $U_j^k$  is the approximation solution of equation (3.5) and let  $\mathcal{E}_j^k = U_j^k - u_j^k$ , then the error equation reads as

$$(3.6) \quad \begin{aligned} & \left( \frac{d_2}{h^2} + \frac{d_1}{2h} \right) \mathcal{E}_{j-1}^{k+1} - \left( \frac{2}{\tau} + \frac{2d_2}{h^2} \right) \mathcal{E}_j^{k+1} + \left( \frac{d_2}{h^2} - \frac{d_1}{2h} \right) \mathcal{E}_{j+1}^{k+1} \\ & - \nu \left[ \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\gamma)} \mathcal{E}_{j-\ell}^{k+1} + \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\gamma)} \mathcal{E}_{j+\ell}^{k+1} \right] = - \left( \frac{d_2}{h^2} + \frac{d_1}{2h} \right) \mathcal{E}_{j-1}^k \\ & - \left( \frac{2}{\tau} - \frac{2d_2}{h^2} \right) \mathcal{E}_j^k - \left( \frac{d_2}{h^2} - \frac{d_1}{2h} \right) \mathcal{E}_{j+1}^k + \nu \left[ \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\gamma)} \mathcal{E}_{j-\ell}^k \right. \\ & \left. + \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\gamma)} \mathcal{E}_{j+\ell}^k \right], \quad j = 1, \dots, M-1, \quad k = 0, 1, \dots, N-1. \end{aligned}$$

Let  $\mathcal{E}_j^k = \xi^k e^{ij\theta}$  be the solution of equation (3.6),  $i = \sqrt{-1}$ , where  $\theta \in [-\pi, \pi]$  is called the phase angle. The stability condition of scheme (3.5) is  $|\xi(\theta)| \leq 1$  for all  $\theta \in [-\pi, \pi]$ .

Substituting  $\varepsilon_j^k = \xi^k e^{ij\theta}$  into (3.6) gives

$$\xi(\theta) = \frac{\left( \frac{2h}{\tau} - \frac{4d_2}{h} \sin^2\left(\frac{\theta}{2}\right) - 2\nu h \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\gamma)} \cos(\ell\theta) \right) - id_1 \sin(\theta)}{\left( \frac{2h}{\tau} + \frac{4d_2}{h} \sin^2\left(\frac{\theta}{2}\right) + 2\nu h \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\gamma)} \cos(\ell\theta) \right) + id_1 \sin(\theta)}.$$

According to Theorem 2.1, we easily obtain  $|\xi(\theta)| \leq 1$ . So scheme (3.5) is unconditionally stable. It is easy to show the convergence order of scheme (3.5) is  $\mathcal{O}(\tau^2 + h^2)$ . In effect, the proof is almost the same as that of [10].  $\square$

**3.2. The 4th-order scheme in space.** Let us first consider the following differential equation

$$(3.7) \quad d_2 \frac{\partial^2 u(x, t)}{\partial x^2} - d_1 \frac{\partial u(x, t)}{\partial x} = g(x, t).$$

Applying the technique presented in [14], we can obtain a fourth-order difference scheme for solving the above equation

$$(3.8) \quad \Delta_4 u(x_j, t) = \tilde{\Delta}_4 g(x_j, t) + \mathcal{O}(h^4).$$

where

$$\Delta_4 = \left( d_2 + \frac{d_1^2 h^2}{12 d_2} \right) \delta_x^2 - d_1 \mu_x \delta_x, \quad \tilde{\Delta}_4 = I + \frac{h^2}{12} \left( \delta_x^2 - \frac{d_1}{d_2} \mu_x \delta_x \right),$$

in which  $I$  is a unit operator.

Combing (3.1) with (3.7) and (3.8) leads to

$$(3.9) \quad \begin{aligned} & \frac{2}{\tau} \tilde{\Delta}_4 \cdot (u(x_j, t_{k+1}) - u(x_j, t_k)) = \Delta_4 \cdot (u(x_j, t_{k+1}) + u(x_j, t_k)) \\ & + d_\alpha \tilde{\Delta}_4 \cdot \left( \frac{\partial^\alpha u(x_j, t_{k+1})}{\partial |x|^\alpha} + \frac{\partial u(x_j, t_k)}{\partial |x|^\alpha} \right) + 2 \tilde{\Delta}_4 \cdot s(x_j, t_{k+\frac{1}{2}}) \\ & + \mathcal{O}(\tau^2 + h^4). \end{aligned}$$

For the Riesz derivative in equation (3.9), we use the fourth-order numerical scheme

$$\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = -\frac{1}{2 \cos(\pi\alpha/2) h^\alpha} \left( \sum_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} u(x - \ell h, t) + \sum_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} u(x + \ell h, t) \right) + \mathcal{O}(h^4).$$

Substituting it into (3.9) and ignoring the truncation error term give

$$\begin{aligned}
& \left( \frac{2a_1}{\tau} - b_1 \right) u_{j-1}^{k+1} + \left( \frac{2a_2}{\tau} - b_2 \right) u_j^{k+1} + \left( \frac{2a_3}{\tau} - b_3 \right) u_{j+1}^{k+1} \\
& + \nu \left[ \sum_{\ell=0}^j \varpi_{4,\ell}^{(\alpha)} \left( a_1 u_{j-\ell-1}^{k+1} + a_2 u_{j-\ell}^{k+1} + a_3 u_{j-\ell+1}^{k+1} \right) \right. \\
& \left. + \sum_{\ell=0}^{M-j} \varpi_{4,\ell}^{(\alpha)} \left( a_1 u_{j+\ell-1}^{k+1} + a_2 u_{j+\ell}^{k+1} + a_3 u_{j+\ell+1}^{k+1} \right) \right] \\
(3.10) \quad & = \left( \frac{2a_1}{\tau} + b_1 \right) u_{j-1}^k + \left( \frac{2a_2}{\tau} + b_2 \right) u_j^k + \left( \frac{2a_3}{\tau} + b_3 \right) u_{j+1}^k \\
& - \nu \left[ \sum_{\ell=0}^j \varpi_{4,\ell}^{(\alpha)} \left( a_1 u_{j-\ell-1}^k + a_2 u_{j-\ell}^k + a_3 u_{j-\ell+1}^k \right) \right. \\
& \left. + \sum_{\ell=0}^{M-j} \varpi_{4,\ell}^{(\alpha)} \left( a_1 u_{j+\ell-1}^k + a_2 u_{j+\ell}^k + a_3 u_{j+\ell+1}^k \right) \right] \\
& + 2a_1 s_{j-1}^{k+\frac{1}{2}} + 2a_2 s_j^{k+\frac{1}{2}} + 2a_3 s_{j+1}^{k+\frac{1}{2}}, j = 1, \dots, M-1, \\
& k = 0, 1, \dots, N-1.
\end{aligned}$$

Here, the parameters in (3.10) are given below,

$$\begin{aligned}
\nu &= \frac{d_\alpha}{2 \cos(\pi\alpha/2) h^\alpha}, \quad a_1 = \frac{1}{12} + \frac{d_1 h}{24d_2}, \quad a_2 = \frac{5}{6}, \quad a_3 = \frac{1}{12} - \frac{d_1 h}{24d_2}, \\
b_1 &= \frac{d_2}{h^2} + \frac{d_1^2}{12d_2} + \frac{d_1}{2h}, \quad b_2 = -2 \left( \frac{d_2}{h^2} + \frac{d_1^2}{12d_2} \right), \quad b_3 = \frac{d_2}{h^2} + \frac{d_1^2}{12d_2} - \frac{d_1}{2h}.
\end{aligned}$$

**Theorem 3.2.** *When  $0 < \alpha \leq 0.8439$ , the numerical scheme (3.10) is unconditionally stable and convergent with order  $\mathcal{O}(\tau^2 + h^4)$ .*

*Proof.* Similar to the above subsection, we can get the error equation of equation (3.10) and let  $\mathcal{E}_j^k = \xi^k e^{ij\theta}$ ,  $\theta \in [-\pi, \pi]$ . Substituting it into the error equation gives

$$\xi(\theta) = \frac{(s_1 - s_2 - s_3\nu) - is_4\nu \sin(\theta)}{(s_1 + s_2 + s_3\nu) + is_4\nu \sin(\theta)}$$

where

$$\begin{aligned}
s_1 &= \frac{2}{\tau} \left( 1 - \frac{1}{3} \sin^2 \left( \frac{\theta}{2} \right) \right), \quad s_2 = 2 \sin^2 \left( \frac{\theta}{2} \right) \left( \frac{2d_2}{h^2} + \frac{d_1^2}{6d_2} \right), \\
s_3 &= 2 \left( 1 - \frac{1}{3} \sin^2 \left( \frac{\theta}{2} \right) \right) \sum_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} \cos(\ell\theta), \quad s_4 = \left( \frac{d_1 h}{6\tau d_2} + \frac{d_1}{h} \right) - \frac{d_1 h}{6d_2} \sum_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} \cos(\ell\theta).
\end{aligned}$$

Note that  $\nu, s_2, s_3 \geq 0$ . It follows from Theorem 2.2 that  $|\xi(\theta)| \leq 1$  if  $0 < \alpha \leq 0.8439$ . So scheme (3.10) is unconditionally stable if  $0 < \alpha \leq 0.8439$ . For such  $\alpha \in (0, 0.8439]$ , the convergence order of scheme (3.10) is  $\mathcal{O}(\tau^2 + h^4)$ .  $\square$

**3.3. The 6th-order scheme in space.** From Taylor expansion, we have

$$(3.11) \quad \frac{\partial u(x_j, t)}{\partial x} = \mu_x \delta_x \left( I - \frac{h^2}{6} \delta_x^2 \right) u(x_j, t) + \frac{h^4}{30} \frac{\partial^5 u(x_j, t)}{\partial x^5} + \mathcal{O}(h^6),$$

$$(3.12) \quad \frac{\partial^2 u(x_j, t)}{\partial x^2} = \delta_x^2 \left( I - \frac{h^2}{12} \delta_x^2 \right) u(x_j, t) + \frac{h^4}{90} \frac{\partial^6 u(x_j, t)}{\partial x^6} + \mathcal{O}(h^6),$$

$$(3.13) \quad \frac{\partial^3 u(x_j, t)}{\partial x^3} = \mu_x \delta_x^3 u(x_j, t) + \mathcal{O}(h^2),$$

$$(3.14) \quad \frac{\partial^4 u(x_j, t)}{\partial x^4} = \delta_x^4 u(x_j, t) + \mathcal{O}(h^2).$$

According to (3.7), one gets

$$(3.15) \quad \frac{\partial^5 u(x_j, t)}{\partial x^5} = \frac{d_1}{d_2} \frac{\partial^4 u(x_j, t)}{\partial x^4} + \frac{1}{d_2} \frac{\partial^3 g(x_j, t)}{\partial x^3}$$

and

$$(3.16) \quad \frac{\partial^6 u(x_j, t)}{\partial x^6} = \frac{d_1^2}{d_2^2} \frac{\partial^4 u(x_j, t)}{\partial x^4} + \frac{1}{d_2} \frac{\partial^4 g(x_j, t)}{\partial x^4} + \frac{d_1}{d_2^2} \frac{\partial^3 g(x_j, t)}{\partial x^3}.$$

From equations (3.11)-(3.16), one has

$$(3.17) \quad \begin{aligned} \frac{\partial u(x_j, t)}{\partial x} = & \mu_x \delta_x \left( I - \frac{h^2}{6} \delta_x^2 \right) u(x_j, t) + \frac{d_1 h^4}{30 d_2} \delta_x^4 u(x_j, t) \\ & + \frac{h^4}{30 d_2} \mu_x \delta_x^3 g(x_j, t) + \mathcal{O}(h^6), \end{aligned}$$

and

$$(3.18) \quad \begin{aligned} \frac{\partial^2 u(x_j, t)}{\partial x^2} = & \delta_x^2 \left( I - \frac{h^2}{12} \delta_x^2 \right) u(x_j, t) + \frac{d_1^2 h^4}{90 d_2^2} \delta_x^4 u(x_j, t) \\ & + \frac{h^4}{90 d_2} \delta_x^4 g(x_j, t) + \frac{h^4 d_1}{90 d_2^2} \mu_x \delta_x^3 g(x_j, t) + \mathcal{O}(h^6). \end{aligned}$$

Substituting (3.17) and (3.18) into (3.7) gives

$$(3.19) \quad \Delta_6 u(x_j, t) = \tilde{\Delta}_6 g(x_j, t) + \mathcal{O}(h^6),$$

where

$$\begin{aligned} \Delta_6 = & d_2 \delta_x^2 \left( I - \frac{h^2}{12} \delta_x^2 \right) - d_1 \mu_x \delta_x \left( I - \frac{h^2}{6} \delta_x^2 \right) - \frac{d_1^2 h^4}{45 d_2} \delta_x^4, \\ \tilde{\Delta}_6 = & I - \frac{h^4}{90} \delta_x^2 \left( \delta_x^2 - \frac{2 d_1}{d_2} \mu_x \delta_x \right). \end{aligned}$$

Combing (3.1) with (3.7) and (3.19), we can obtain

$$(3.20) \quad \begin{aligned} & \frac{2}{\tau} \tilde{\Delta}_6 \cdot (u(x_j, t_{k+1}) - u(x_j, t_k)) = \Delta_6 \cdot (u(x_j, t_{k+1}) + u(x_j, t_k)) \\ & + d_\gamma \tilde{\Delta}_6 \cdot \left( \frac{\partial u(x_j, t_{k+1})}{\partial |x|^\gamma} + \frac{\partial u(x_j, t_k)}{\partial |x|^\gamma} \right) + 2 \tilde{\Delta}_6 \cdot s(x_j, t_{k+\frac{1}{2}}) \\ & + \mathcal{O}(\tau^2 + h^6). \end{aligned}$$



For the Riesz derivative, we apply the sixth-order numerical scheme

$$\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = -\frac{1}{2 \cos(\pi\alpha/2) h^\alpha} \left( \sum_{\ell=0}^{\infty} \varpi_{6,\ell}^{(\alpha)} f(x - \ell h) + \sum_{\ell=0}^{\infty} \varpi_{6,\ell}^{(\alpha)} f(x + \ell h) \right) + \mathcal{O}(h^6).$$

Substitution in (3.20) gives

$$\begin{aligned} & \left( \frac{2c_1}{\tau} + e_1 \right) u_{j-2}^{k+1} + \left( \frac{2c_2}{\tau} + e_2 \right) u_{j-1}^{k+1} + \left( \frac{2c_3}{\tau} + e_3 \right) u_j^{k+1} \\ & + \left( \frac{2c_4}{\tau} + e_4 \right) u_{j+1}^{k+1} + \left( \frac{2c_5}{\tau} + e_5 \right) u_{j+2}^{k+1} + \nu \left[ \sum_{\ell=0}^j \varpi_{6,\ell}^{(\alpha)} \right. \\ & \left( c_1 u_{j-l-2}^{k+1} + c_2 u_{j-l-1}^{k+1} + c_3 u_{j-l}^{k+1} + c_4 u_{j-l+1}^{k+1} + c_5 u_{j-l+2}^{k+1} \right) \\ & + \sum_{\ell=0}^{M-j} \varpi_{6,\ell}^{(\alpha)} \left( c_1 u_{j+l-2}^{k+1} + c_2 u_{j+l-1}^{k+1} + c_3 u_{j+l}^{k+1} + c_4 u_{j+l+1}^{k+1} \right. \\ & \left. \left. + c_5 u_{j+l+2}^{k+1} \right) \right] = \left( \frac{2c_1}{\tau} - e_1 \right) u_{j-2}^k + \left( \frac{2c_2}{\tau} - e_2 \right) u_{j-1}^k \\ (3.21) \quad & + \left( \frac{2c_3}{\tau} - e_3 \right) u_j^k + \left( \frac{2c_4}{\tau} - e_4 \right) u_{j+1}^k + \left( \frac{2c_5}{\tau} - e_5 \right) u_{j+2}^k \\ & - \nu \left[ \sum_{\ell=0}^j \varpi_{6,\ell}^{(\alpha)} \left( c_1 u_{j-l-2}^k + c_2 u_{j-l-1}^k + c_3 u_{j-l}^{k+1} + c_4 u_{j-l+1}^k \right. \right. \\ & \left. \left. + c_5 u_{j-l+2}^k \right) + \sum_{\ell=0}^{M-j} \varpi_{6,\ell}^{(\alpha)} \left( c_1 u_{j+l-2}^k + c_2 u_{j+l-1}^k + c_3 u_{j+l}^k \right. \right. \\ & \left. \left. + c_4 u_{j+l+1}^k + c_5 u_{j+l+2}^k \right) \right] + 2c_1 f_{j-2}^{k+\frac{1}{2}} + 2c_2 f_{j-1}^{k+\frac{1}{2}} \\ & + 2c_3 f_j^{k+\frac{1}{2}} + 2c_4 f_{j+1}^{k+\frac{1}{2}} + 2c_5 f_{j+2}^{k+\frac{1}{2}}, \quad j = 2, \dots, M-2, \\ & k = 0, 1, \dots, N-1. \end{aligned}$$

Here

$$\begin{aligned} \nu &= \frac{d_\alpha}{2 \cos(\pi\alpha/2) h^\alpha}, \quad c_1 = -\frac{1}{90} \left( 1 + \frac{d_1 h}{d_2} \right), \quad c_2 = \frac{1}{90} \left( 4 + \frac{2d_1 h}{d_2} \right), \\ c_3 &= \frac{14}{15}, \quad c_4 = \frac{1}{90} \left( 4 - \frac{2d_1 h}{d_2} \right), \quad c_5 = -\frac{1}{90} \left( 1 - \frac{d_1 h}{d_2} \right), \\ e_1 &= \frac{d_2}{12h^2} + \frac{d_1}{12h} + \frac{d_1^2}{45d_2}, \quad e_2 = -\left( \frac{4d_2}{3h^2} + \frac{2d_1}{3h} + \frac{4d_1^2}{45d_2} \right), \\ e_3 &= \frac{5d_2}{2h^2} + \frac{2d_1^2}{15d_2}, \quad e_4 = -\left( \frac{4d_2}{3h^2} - \frac{2d_1}{3h} + \frac{4d_1^2}{45d_2} \right), \quad e_5 = \frac{d_2}{12h^2} - \frac{d_1}{12h} + \frac{d_1^2}{45d_2}. \end{aligned}$$

**Theorem 3.3.** *The numerical scheme (3.21) is unconditionally stable and convergent with order  $\mathcal{O}(\tau^2 + h^6)$ .*

*Proof.* Let  $\mathcal{E}_j^k = \xi^k e^{ij\theta}$ ,  $\theta \in [-\pi, \pi]$ . Substituting it into the error equation of the equation (3.21)

$$\begin{aligned}
& \left( \frac{2c_1}{\tau} + e_1 \right) \mathcal{E}_{j-2}^{k+1} + \left( \frac{2c_2}{\tau} + e_2 \right) \mathcal{E}_{j-1}^{k+1} + \left( \frac{2c_3}{\tau} + e_3 \right) \mathcal{E}_j^{k+1} + \left( \frac{2c_4}{\tau} + e_4 \right) \mathcal{E}_{j+1}^{k+1} \\
& + \left( \frac{2c_5}{\tau} + e_5 \right) \mathcal{E}_{j+2}^{k+1} + \nu \left[ \sum_{\ell=0}^j \varpi_{6,\ell}^{(\alpha)} \left( c_1 \mathcal{E}_{j-\ell-2}^{k+1} + c_2 \mathcal{E}_{j-\ell-1}^{k+1} + c_3 \mathcal{E}_{j-\ell}^{k+1} + c_4 \mathcal{E}_{j-\ell+1}^{k+1} \right. \right. \\
& \left. \left. + c_5 \mathcal{E}_{j-\ell+2}^{k+1} \right) + \sum_{\ell=0}^{M-j} \varpi_{6,\ell}^{(\alpha)} \left( c_1 \mathcal{E}_{j+\ell-2}^{k+1} + c_2 \mathcal{E}_{j+\ell-1}^{k+1} + c_3 \mathcal{E}_{j+\ell}^{k+1} + c_4 \mathcal{E}_{j+\ell+1}^{k+1} + c_5 \mathcal{E}_{j+\ell+2}^{k+1} \right) \right] \\
& = \left( \frac{2c_1}{\tau} - e_1 \right) \mathcal{E}_{j-2}^k + \left( \frac{2c_2}{\tau} - e_2 \right) \mathcal{E}_{j-1}^k + \left( \frac{2c_3}{\tau} - e_3 \right) \mathcal{E}_j^k + \left( \frac{2c_4}{\tau} - e_4 \right) \mathcal{E}_{j+1}^k \\
& + \left( \frac{2c_5}{\tau} - e_5 \right) \mathcal{E}_{j+2}^k - \nu \left[ \sum_{\ell=0}^j \varpi_{6,\ell}^{(\alpha)} \left( c_1 \mathcal{E}_{j-\ell-2}^k + c_2 \mathcal{E}_{j-\ell-1}^k + c_3 \mathcal{E}_{j-\ell}^{k+1} + c_4 \mathcal{E}_{j-\ell+1}^k \right. \right. \\
& \left. \left. + c_5 \mathcal{E}_{j-\ell+2}^k \right) + \sum_{\ell=0}^{M-j} \varpi_{6,\ell}^{(\alpha)} \left( c_1 \mathcal{E}_{j+\ell-2}^k + c_2 \mathcal{E}_{j+\ell-1}^k + c_3 \mathcal{E}_{j+\ell}^k + c_4 \mathcal{E}_{j+\ell+1}^k + c_5 \mathcal{E}_{j+\ell+2}^k \right) \right], \\
& j = 2, \dots, M-2, \quad k = 0, 1, \dots, N-1,
\end{aligned}$$

leads to

$$\xi(\theta) = \frac{(w_1 - w_2 - w_3\nu) - iw_4 \sin(\theta)}{(w_1 + w_2 + w_3\nu) + iw_4 \sin(\theta)}.$$

Here

$$\begin{aligned}
w_1 &= \frac{2}{45\tau} \left[ 45 - 8 \sin^4 \left( \frac{\theta}{2} \right) \right], \quad w_2 = \frac{2d_2}{3h^2} (7 - \cos \theta) \sin^2 \left( \frac{\theta}{2} \right) + \frac{16d_1^2}{45d_2} \sin^4 \left( \frac{\theta}{2} \right), \\
w_3 &= \frac{2}{45} \left[ 45 - 8 \sin^4 \left( \frac{\theta}{2} \right) \right] \sum_{\ell=0}^{\infty} \varpi_{6,\ell}^{(\alpha)} \cos(\ell\theta), \\
w_4 &= -\frac{8d_1h}{45\tau d_2} \sin^2 \left( \frac{\theta}{2} \right) + \frac{d_1}{3h} (4 - \cos \theta) + \frac{8d_1h\nu}{45d_2} \sin^2 \left( \frac{\theta}{2} \right) \sum_{\ell=0}^{\infty} \varpi_{6,\ell}^{(\alpha)} \cos(\ell\theta).
\end{aligned}$$

It is clear that  $\nu, w_2, w_3 \geq 0$ . By using Theorem 2.1, one also has

$$|\xi(\theta)| \leq 1.$$

So scheme (3.21) is unconditionally stable. It can be shown that the convergence order of scheme (3.21) for equation (1.1) is  $\mathcal{O}(\tau^2 + h^6)$ .  $\square$

In the next section, we present several numerical examples.

#### 4. NUMERICAL EXAMPLES

Firstly, we test the higher-order schemes for Riesz derivatives.

**Example 1.** Consider the function  $f_p(x) = x^p(1-x)^p$ ,  $x \in [0, 1]$ ,  $p = 2, 3, 4, 5, 6$ .

The Riesz derivative of the above function is analytically expressed as

$$\frac{\partial^\alpha f_p(x)}{\partial |x|^\alpha} = -\frac{1}{2 \cos(\pi\alpha/2)} \sum_{\ell=0}^p (-1)^\ell \frac{p!(p+\ell)!}{\ell!(p-\ell)!\Gamma(p+\ell+1-\alpha)} [x^{p+\ell-\alpha} + (1-x)^{p+\ell-\alpha}].$$

We numerically solve  $f_p(x)$  by using numerical scheme (2.3). The numerical results are presented in Tables 1-5. From these tables, the experimental orders are in line with the theoretical orders  $p$  ( $p = 2, 3, 4, 5, 6$ ).

TABLE 1. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with  $p = 2$ .

$\alpha$	$h$	the absolute error	the convergence order
0.2	$\frac{1}{20}$	2.381267e-004	—
	$\frac{1}{40}$	5.900964e-005	2.0127
	$\frac{1}{80}$	1.460639e-005	2.0144
	$\frac{1}{160}$	3.628491e-006	2.0092
	$\frac{1}{320}$	9.039358e-007	2.0051
0.4	$\frac{1}{20}$	7.097814e-004	—
	$\frac{1}{40}$	1.696639e-004	2.0647
	$\frac{1}{80}$	4.123703e-005	2.0407
	$\frac{1}{160}$	1.014980e-005	2.0225
	$\frac{1}{320}$	2.516782e-006	2.0118
0.6	$\frac{1}{20}$	1.638369e-003	—
	$\frac{1}{40}$	3.728453e-004	2.1356
	$\frac{1}{80}$	8.824023e-005	2.0791
	$\frac{1}{160}$	2.141683e-005	2.0427
	$\frac{1}{320}$	5.272483e-006	2.0222
0.8	$\frac{1}{20}$	3.782418e-003	—
	$\frac{1}{40}$	7.826735e-004	2.2728
	$\frac{1}{80}$	1.747182e-004	2.1634
	$\frac{1}{160}$	4.102708e-005	2.0904
	$\frac{1}{320}$	9.923174e-006	2.0477

Next we test the numerical schemes for the equations which have the form of equ. (1.1).

**Example 2.** Consider the following equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\partial u(x, t)}{\partial x} + \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + s(x, t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1,$$

in which

$$\begin{aligned}
f(x, t) = & \exp(t)x^4(1-x)^4(x^4 + 10x^3 - 149x^2 + 138x - 30) + \frac{1}{2}\exp(t)\sec\left(\frac{\pi}{2}\alpha\right) \\
& \left\{ \frac{\Gamma(7)}{\Gamma(7-\alpha)} [x^{6-\alpha} + (1-x)^{6-\alpha}] - \frac{6\Gamma(8)}{\Gamma(8-\alpha)} [x^{7-\alpha} + (1-x)^{7-\alpha}] \right. \\
& + \frac{15\Gamma(9)}{\Gamma(9-\alpha)} [x^{8-\alpha} + (1-x)^{8-\alpha}] - \frac{20\Gamma(10)}{\Gamma(10-\alpha)} [x^{9-\alpha} + (1-x)^{9-\alpha}] \\
& + \frac{15\Gamma(11)}{\Gamma(11-\alpha)} [x^{10-\alpha} + (1-x)^{10-\alpha}] - \frac{6\Gamma(12)}{\Gamma(12-\alpha)} [x^{11-\alpha} + (1-x)^{11-\alpha}] \\
& \left. + \frac{\Gamma(13)}{\Gamma(13-\alpha)} [x^{12-\alpha} + (1-x)^{12-\alpha}] \right\}.
\end{aligned}$$

Its analytical solution is  $u(x, t) = \exp(t)x^6(1-x)^6$  and satisfy the corresponding initial and boundary values conditions.

TABLE 2. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with  $p = 3$ .

$\alpha$	$h$	the absolute error	the convergence order
0.2	$\frac{1}{40}$	3.146678e-007	—
	$\frac{1}{60}$	1.576085e-007	1.7052
	$\frac{1}{80}$	7.991483e-008	2.3608
	$\frac{1}{100}$	4.501080e-008	2.5726
	$\frac{1}{120}$	2.761993e-008	2.6786
0.4	$\frac{1}{40}$	4.349687e-006	—
	$\frac{1}{60}$	1.491902e-006	2.6391
	$\frac{1}{80}$	6.709309e-007	2.7779
	$\frac{1}{100}$	3.560502e-007	2.8394
	$\frac{1}{120}$	2.108270e-007	2.8742
0.6	$\frac{1}{40}$	2.194879e-005	—
	$\frac{1}{60}$	6.996810e-006	2.8196
	$\frac{1}{80}$	3.050074e-006	2.8861
	$\frac{1}{100}$	1.590859e-006	2.9169
	$\frac{1}{120}$	9.316704e-007	2.9347
0.8	$\frac{1}{40}$	1.067572e-004	—
	$\frac{1}{60}$	3.282279e-005	2.9088
	$\frac{1}{80}$	1.407258e-005	2.9439
	$\frac{1}{100}$	7.270149e-006	2.9598
	$\frac{1}{120}$	4.231299e-006	2.9688

We solve this problem with the numerical schemes (3.5) and (3.10) for different values of  $\tau$ ,  $h$  and  $\alpha$ . The absolute error, temporal and spatial convergence orders are listed in Tables 6 and 7, which display that the numerical results are in line with our theoretical analysis.

In the following, we give a slightly different example.

**Example 3.** Consider the following equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} - 2 \frac{\partial u(x, t)}{\partial x} + \alpha^2 \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + s(x, t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1,$$

TABLE 3. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with  $p = 4$ .

$\alpha$	$h$	the absolute error	the convergence order
0.2	$\frac{1}{20}$	3.254967e-006	—
	$\frac{1}{25}$	1.421712e-006	3.7121
	$\frac{1}{30}$	7.061638e-007	3.8381
	$\frac{1}{35}$	3.863551e-007	3.9123
	$\frac{1}{40}$	2.279637e-007	3.9509
0.4	$\frac{1}{20}$	1.307893e-005	—
	$\frac{1}{25}$	5.433994e-006	3.9362
	$\frac{1}{30}$	2.610192e-006	4.0217
	$\frac{1}{35}$	1.395202e-006	4.0635
	$\frac{1}{40}$	8.089264e-007	4.0821
0.6	$\frac{1}{20}$	4.165885e-005	—
	$\frac{1}{25}$	1.647646e-005	4.1569
	$\frac{1}{30}$	7.642179e-006	4.2137
	$\frac{1}{35}$	3.980841e-006	4.2309
	$\frac{1}{40}$	2.261840e-006	4.2336
0.8	$\frac{1}{20}$	1.466022e-004	—
	$\frac{1}{25}$	5.460563e-005	4.4258
	$\frac{1}{30}$	2.420431e-005	4.4625
	$\frac{1}{35}$	1.216174e-005	4.4647
	$\frac{1}{40}$	6.707129e-006	4.4568

in which

$$\begin{aligned}
s(x, t) = & x^6(1-x)^6 [\cos(t)(x^4 - 2x^3 + x^2) + \sin(t)(32x^3 - 288x^2 + 256x - 56)] \\
& + \frac{\alpha^2}{2} \sin(t) \sec\left(\frac{\pi}{2}\alpha\right) \left\{ \frac{\Gamma(9)}{\Gamma(9-\alpha)} [x^{8-\alpha} + (1-x)^{8-\alpha}] \right. \\
& - \frac{8\Gamma(10)}{\Gamma(10-\alpha)} [x^{9-\alpha} + (1-x)^{9-\alpha}] + \frac{28\Gamma(11)}{\Gamma(11-\alpha)} [x^{10-\alpha} + (1-x)^{10-\alpha}] \\
& - \frac{56\Gamma(12)}{\Gamma(12-\alpha)} [x^{11-\alpha} + (1-x)^{11-\alpha}] + \frac{70\Gamma(13)}{\Gamma(13-\alpha)} [x^{12-\alpha} + (1-x)^{12-\alpha}] \\
& - \frac{56\Gamma(14)}{\Gamma(14-\alpha)} [x^{13-\alpha} + (1-x)^{13-\alpha}] + \frac{28\Gamma(15)}{\Gamma(15-\alpha)} [x^{14-\alpha} + (1-x)^{14-\alpha}] \\
& \left. - \frac{8\Gamma(16)}{\Gamma(16-\alpha)} [x^{15-\alpha} + (1-x)^{15-\alpha}] + \frac{\Gamma(17)}{\Gamma(17-\alpha)} [x^{16-\alpha} + (1-x)^{16-\alpha}] \right\}.
\end{aligned}$$

Its exact solution is  $u(x, t) = \sin(t)x^8(1-x)^8$  and satisfy the accordingly initial and boundary values conditions.

The absolute error, temporal and spatial convergence orders are listed in Table 8 by numerical scheme (3.21). The numerical results agree with the theoretical results.

TABLE 4. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with  $p = 5$ .

$\alpha$	$h$	the absolute error	the convergence order
0.2	$\frac{1}{80}$	3.254967e-006	—
	$\frac{1}{100}$	1.421712e-006	3.7121
	$\frac{1}{120}$	7.061638e-007	3.8381
	$\frac{1}{140}$	3.863551e-007	3.9123
	$\frac{1}{160}$	2.279637e-007	3.9509
0.4	$\frac{1}{80}$	8.731739e-010	—
	$\frac{1}{100}$	3.348011e-010	4.2959
	$\frac{1}{120}$	1.473288e-010	4.5023
	$\frac{1}{140}$	7.232096e-011	4.6160
	$\frac{1}{160}$	3.867341e-011	4.6877
0.6	$\frac{1}{80}$	5.385482e-009	—
	$\frac{1}{100}$	1.900398e-009	4.6680
	$\frac{1}{120}$	7.985621e-010	4.7554
	$\frac{1}{140}$	3.806168e-010	4.8071
	$\frac{1}{160}$	1.994064e-010	4.8412
0.8	$\frac{1}{80}$	3.005433e-008	—
	$\frac{1}{100}$	1.023908e-008	4.8256
	$\frac{1}{120}$	4.211445e-009	4.8727
	$\frac{1}{140}$	1.978515e-009	4.9008
	$\frac{1}{160}$	1.025811e-009	4.9192

## 5. CONCLUSIONS

In this paper, we construct high-order (from 2nd-order to 6th-order) numerical schemes to approximate the Riesz derivatives. Next, we develop three kinds of difference schemes for the Riesz space fractional turbulent diffusion equation. The stability of the derived numerical algorithms are shown by the Fourier method. The convergence orders are  $\mathcal{O}(\tau^2 + h^2)$ ,  $\mathcal{O}(\tau^2 + h^4)$  and  $\mathcal{O}(\tau^2 + h^6)$ , respectively. Finally, numerical results confirm the theoretical analysis.

### *Appendix A*

*Proof.* (1) Direct calculations can finish it so we omit the details.  
(2) See [11] for details.

TABLE 5. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with  $p = 6$ .

$\alpha$	$h$	the absolute error	the convergence order
0.2	$\frac{1}{20}$	3.783855e-008	—
	$\frac{1}{40}$	2.310553e-009	4.0335
	$\frac{1}{80}$	4.258651e-011	5.7617
	$\frac{1}{160}$	6.690552e-013	5.9921
	$\frac{1}{320}$	1.035841e-014	6.0133
0.4	$\frac{1}{20}$	3.116503e-007	—
	$\frac{1}{40}$	1.031745e-008	4.9168
	$\frac{1}{80}$	1.651582e-010	5.9651
	$\frac{1}{160}$	2.418476e-012	6.0936
	$\frac{1}{320}$	3.582209e-014	6.0771
0.6	$\frac{1}{20}$	1.564617e-006	—
	$\frac{1}{40}$	3.647148e-008	5.4229
	$\frac{1}{80}$	5.062291e-010	6.1708
	$\frac{1}{160}$	6.799919e-012	6.2181
	$\frac{1}{320}$	9.539537e-014	6.1555
0.8	$\frac{1}{20}$	8.311643e-006	—
	$\frac{1}{40}$	1.432632e-007	5.8584
	$\frac{1}{80}$	1.663072e-009	6.4287
	$\frac{1}{160}$	1.942938e-011	6.4195
	$\frac{1}{320}$	2.433601e-013	6.3190

(3) Now we show the case  $\alpha \in (1, 2)$ . We firstly show that  $\varpi_{2,j}^{(\alpha)} > 0$  for  $\ell \geq 5$ . For convenience, denote  $\alpha = 1 + \gamma$ , where  $0 < \gamma < 1$ . Lengthy calculations give

$$\begin{aligned}
\varpi_{2,\ell}^{(\alpha)} &= \left(\frac{3}{2}\right)^{1+\gamma} \sum_{m=0}^{\ell} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1+\gamma)} \varpi_{1,\ell-m}^{(1+\gamma)} \\
&= \left(\frac{3}{2}\right)^{1+\gamma} \left[ \varpi_{1,0}^{(1+\gamma)} \varpi_{1,\ell}^{(1+\gamma)} + \frac{1}{3} \varpi_{1,1}^{(1+\gamma)} \varpi_{1,\ell-1}^{(1+\gamma)} + \frac{1}{9} \varpi_{1,2}^{(1+\gamma)} \varpi_{1,\ell-2}^{(1+\gamma)} \right. \\
&\quad \left. + \left(\frac{1}{3}\right)^{\ell} \varpi_{1,0}^{(1+\gamma)} \varpi_{1,\ell}^{(1+\gamma)} + \left(\frac{1}{3}\right)^{\ell-1} \varpi_{1,1}^{(1+\gamma)} \varpi_{1,\ell-1}^{(1+\gamma)} \right] \\
&\quad + \left(\frac{3}{2}\right)^{1+\gamma} \sum_{m=3}^{\ell-2} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1+\gamma)} \varpi_{1,\ell-m}^{(1+\gamma)} \\
&= \left(\frac{3}{2}\right)^{1+\gamma} \left[ 1 - \frac{\gamma+1}{3} \frac{\ell}{(\ell-2-\gamma)} + \frac{\gamma(\gamma+1)}{18} \frac{\ell(\ell-1)}{(\ell-2-\gamma)(\ell-3-\gamma)} \right. \\
&\quad \left. + \left(\frac{1}{3}\right)^{\ell} \left( 1 - \frac{3\ell(\gamma+1)}{(\ell-2-\gamma)} \right) \right] \varpi_{1,\ell}^{(1+\gamma)} + \left(\frac{3}{2}\right)^{1+\gamma} \sum_{m=3}^{\ell-2} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1+\gamma)} \varpi_{1,\ell-m}^{(1+\gamma)} \\
&\geq \left(\frac{3}{2}\right)^{1-\alpha} \left[ \frac{244}{243} - \frac{28(\gamma+1)}{81} \frac{\ell}{(\ell-2-\gamma)} + \frac{\gamma(\gamma+1)}{18} \frac{\ell(\ell-1)}{(\ell-2-\gamma)^2} \right] \varpi_{1,\ell}^{(1+\gamma)} \\
&\quad + \left(\frac{3}{2}\right)^{1+\gamma} \sum_{m=3}^{\ell-2} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1+\gamma)} \varpi_{1,\ell-m}^{(1+\gamma)}, \quad \ell \geq 5.
\end{aligned}$$

TABLE 6. The absolute errors, temporal and spatial convergence orders of Example 2 by difference scheme (3.5).

$\alpha$		the maximum errors	temporal convergence orders	spatial convergence orders
0.2	$h = \frac{1}{10}, \tau = \frac{1}{10}$	2.581219e-005	—	—
	$h = \frac{1}{20}, \tau = \frac{1}{20}$	6.217660e-006	2.0536	2.0536
	$h = \frac{1}{40}, \tau = \frac{1}{40}$	1.536085e-006	2.0171	2.0171
	$h = \frac{1}{80}, \tau = \frac{1}{80}$	3.844480e-007	1.9984	1.9984
0.3	$h = \frac{1}{10}, \tau = \frac{1}{10}$	2.514418e-005	—	—
	$h = \frac{1}{20}, \tau = \frac{1}{20}$	6.061411e-006	2.0525	2.0525
	$h = \frac{1}{40}, \tau = \frac{1}{40}$	1.498825e-006	2.0158	2.0158
	$h = \frac{1}{80}, \tau = \frac{1}{80}$	3.755193e-007	1.9969	1.9969
0.4	$h = \frac{1}{10}, \tau = \frac{1}{10}$	2.416676e-005	—	—
	$h = \frac{1}{20}, \tau = \frac{1}{20}$	5.842791e-006	2.0483	2.0483
	$h = \frac{1}{40}, \tau = \frac{1}{40}$	1.448271e-006	2.0123	2.0123
	$h = \frac{1}{80}, \tau = \frac{1}{80}$	3.636282e-007	1.9938	1.9938
0.5	$h = \frac{1}{10}, \tau = \frac{1}{10}$	2.270557e-005	—	—
	$h = \frac{1}{20}, \tau = \frac{1}{20}$	5.532646e-006	2.0370	2.0370
	$h = \frac{1}{40}, \tau = \frac{1}{40}$	1.379247e-006	2.0041	2.0041
	$h = \frac{1}{80}, \tau = \frac{1}{80}$	3.477777e-007	1.9876	1.9876
0.6	$h = \frac{1}{10}, \tau = \frac{1}{10}$	2.043415e-005	—	—
	$h = \frac{1}{20}, \tau = \frac{1}{20}$	5.079755e-006	2.0082	2.0082
	$h = \frac{1}{40}, \tau = \frac{1}{40}$	1.283408e-006	1.9848	1.9848
	$h = \frac{1}{80}, \tau = \frac{1}{80}$	3.264917e-007	1.9749	1.9749
0.7	$h = \frac{1}{10}, \tau = \frac{1}{10}$	1.664449e-005	—	—
	$h = \frac{1}{20}, \tau = \frac{1}{20}$	4.378341e-006	1.9266	1.9266
	$h = \frac{1}{40}, \tau = \frac{1}{40}$	1.145019e-006	1.9350	1.9350
	$h = \frac{1}{80}, \tau = \frac{1}{80}$	2.972789e-007	1.9455	1.9455

Let

$$F(x, \gamma) = \frac{244}{243} - \frac{28(\gamma + 1)}{81} \frac{x}{(x - 2 - \gamma)} + \frac{\gamma(\gamma + 1)}{18} \frac{x(x - 1)}{(x - 2 - \gamma)^2}, \quad x \geq 5,$$

and

$$\begin{aligned} G(x, \gamma) &= 486(x - 2 - \gamma)^2 F(x, \gamma) \\ &= 488(x - 2 - \gamma)^2 - 168(\gamma + 1)x(x - 2 - \gamma) + 27\gamma(\gamma + 1)x(x - 1), \quad x \geq 5. \end{aligned}$$

Then

$$G_x(x, \gamma) = 976(x - 2 - \gamma) - 168(\gamma + 1)(2x - 2 - \gamma) + 27\gamma(\gamma + 1)(2x - 1)$$

and

$$G_{xx}(x, \gamma) = 54\gamma^2 - 282\gamma + 640.$$

Obviously,

$$G_{xx}(x, \gamma) \geq 0 \quad \text{for } 0 < \gamma < 1,$$



TABLE 7. The maximum errors, temporal and spatial convergence orders of Example 2 by difference scheme (3.10).

$\alpha$		the maximum errors	temporal convergence orders	spatial convergence orders
0.2	$h = \frac{1}{4}, \tau = \frac{1}{4}$	1.151043e-004	—	—
	$h = \frac{1}{8}, \tau = \frac{1}{16}$	4.361384e-006	2.3610	4.7220
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	2.346706e-007	2.1081	4.2161
	$h = \frac{1}{32}, \tau = \frac{1}{256}$	2.036847e-008	1.7631	3.5262
0.3	$h = \frac{1}{4}, \tau = \frac{1}{4}$	1.128702e-004	—	—
	$h = \frac{1}{8}, \tau = \frac{1}{16}$	4.362181e-006	2.3468	4.6935
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	2.461975e-007	2.0736	4.1472
	$h = \frac{1}{32}, \tau = \frac{1}{256}$	2.396154e-008	1.6805	3.3610
0.4	$h = \frac{1}{4}, \tau = \frac{1}{4}$	1.088961e-004	—	—
	$h = \frac{1}{8}, \tau = \frac{1}{16}$	4.433621e-006	2.3091	4.6183
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	2.870612e-007	1.9746	3.9491
	$h = \frac{1}{32}, \tau = \frac{1}{256}$	2.910816e-008	1.6509	3.3019
0.5	$h = \frac{1}{4}, \tau = \frac{1}{4}$	1.018053e-004	—	—
	$h = \frac{1}{8}, \tau = \frac{1}{16}$	4.654112e-006	2.2256	4.4512
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	3.607253e-007	1.8447	3.6895
	$h = \frac{1}{32}, \tau = \frac{1}{256}$	3.674011e-008	1.6478	3.2955
0.6	$h = \frac{1}{4}, \tau = \frac{1}{4}$	8.897936e-005	—	—
	$h = \frac{1}{8}, \tau = \frac{1}{16}$	5.201584e-006	2.0482	4.0964
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	4.980729e-007	1.6923	3.3845
	$h = \frac{1}{32}, \tau = \frac{1}{256}$	4.862623e-008	1.6783	3.3566
0.7	$h = \frac{1}{4}, \tau = \frac{1}{4}$	6.521192e-005	—	—
	$h = \frac{1}{8}, \tau = \frac{1}{16}$	6.540139e-006	1.6588	3.3177
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	7.724068e-007	1.5410	3.0819
	$h = \frac{1}{32}, \tau = \frac{1}{256}$	6.867979e-008	1.7457	3.4914

it immediately follows that  $G_x(x, \gamma)$  is an increasing function and  $G_x(x, \gamma) \geq G_x(5, \gamma)$  for  $x \geq 5$ .

Note that

$$G_x(5, \gamma) = 411\gamma^2 - 1909\gamma + 1585 > 0, \quad 0 < \gamma < 1.$$

Hence  $G(x, \gamma)$  is an increasing function too, and  $G(x, \gamma) \geq G(5, \gamma)$  if  $x \geq 5$ . Simple calculations yields

$$G(5, \gamma) = 1868\gamma^2 - 3068\gamma + 1872 \geq G_{\min}(5, \gamma) = G\left(5, \frac{767}{934}\right) = 612\frac{131}{467},$$

so,  $G(x, \gamma) \geq 0$ . Therefore, the following inequality holds

$$F(x, \gamma) = \frac{G(x, \gamma)}{486(x-2-\gamma)^2} \geq 0,$$

which means  $\varpi_{2,j}^{(1+\gamma)} > 0$  for  $\ell \geq 5$ . Here we used the positivity of the coefficient  $\varpi_{1,\ell}^{(1+\gamma)} (\ell \geq 2)$ .

TABLE 8. The maximum errors, temporal and spatial convergence orders of Example 3 by difference scheme (3.21).

$\alpha$		the maximum errors	temporal convergence orders	spatial convergence orders
0.2	$h = \frac{1}{8}, \tau = \frac{1}{8}$	1.360207e-007	—	—
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	2.071201e-009	2.0124	6.0372
	$h = \frac{1}{32}, \tau = \frac{1}{512}$	3.348089e-011	1.9837	5.9510
	$h = \frac{1}{64}, \tau = \frac{1}{4096}$	5.235085e-013	1.9997	5.9990
0.3	$h = \frac{1}{8}, \tau = \frac{1}{8}$	1.356431e-007	—	—
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	2.092867e-009	2.0061	6.0182
	$h = \frac{1}{32}, \tau = \frac{1}{512}$	3.254863e-011	2.0022	6.0067
	$h = \frac{1}{64}, \tau = \frac{1}{4096}$	4.855887e-013	2.0222	6.0667
0.4	$h = \frac{1}{8}, \tau = \frac{1}{8}$	1.348600e-007	—	—
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	2.146379e-009	1.9911	5.9734
	$h = \frac{1}{32}, \tau = \frac{1}{512}$	2.972961e-011	2.0580	6.1739
	$h = \frac{1}{64}, \tau = \frac{1}{4096}$	3.852828e-013	2.0899	6.2698
0.5	$h = \frac{1}{8}, \tau = \frac{1}{8}$	1.335205e-007	—	—
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	2.246228e-009	1.9645	5.8934
	$h = \frac{1}{32}, \tau = \frac{1}{512}$	2.200601e-011	2.2245	6.6735
	$h = \frac{1}{64}, \tau = \frac{1}{4096}$	2.816274e-013	2.0960	6.2880
0.6	$h = \frac{1}{8}, \tau = \frac{1}{8}$	1.322258e-007	—	—
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	2.372915e-009	1.9334	5.8002
	$h = \frac{1}{32}, \tau = \frac{1}{512}$	1.827817e-011	2.3401	7.0204
	$h = \frac{1}{64}, \tau = \frac{1}{4096}$	4.506292e-013	1.7807	5.3420
0.7	$h = \frac{1}{8}, \tau = \frac{1}{8}$	1.357968e-007	—	—
	$h = \frac{1}{16}, \tau = \frac{1}{64}$	3.805230e-009	1.7191	5.1573
	$h = \frac{1}{32}, \tau = \frac{1}{512}$	6.671019e-011	1.9446	5.8339
	$h = \frac{1}{64}, \tau = \frac{1}{4096}$	1.819328e-012	1.7321	5.1964

Next, we show that  $\varpi_{2,\ell}^{(\alpha)} > \varpi_{2,\ell+1}^{(\alpha)}$  for  $\ell \geq 5$ . Note that

$$\begin{aligned}
\varpi_{2,\ell}^{(\alpha)} - \varpi_{2,\ell+1}^{(\alpha)} &= (2+\gamma) \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^\ell \sum_{\ell_1=0}^{\ell} \frac{3^{\ell_1}}{\ell_1+1} \varpi_{1,\ell_1}^{(1+\gamma)} \varpi_{1,\ell-\ell_1}^{(1+\gamma)} \\
&\quad - \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^{\ell+1} \left(1 - \frac{2+\gamma}{\ell+1}\right) \varpi_{1,\ell}^{(1+\gamma)} \\
&= \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^\ell \left[ (2+\gamma) \sum_{\ell_1=0}^{\ell} \frac{3^{\ell_1}}{\ell_1+1} \varpi_{1,\ell_1}^{(1+\gamma)} \varpi_{1,\ell-\ell_1}^{(1+\gamma)} - \frac{1}{3} \varpi_{1,\ell}^{(1+\gamma)} \right] \\
&\quad + \frac{2+\gamma}{\ell+1} \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^{\ell+1} \varpi_{1,\ell}^{(1+\gamma)} \\
&\geq (2+\gamma) \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^\ell \left[ \sum_{\ell_1=0}^{\ell} \frac{3^{\ell_1}}{\ell_1+1} \varpi_{1,\ell_1}^{(1+\gamma)} \varpi_{1,\ell-\ell_1}^{(1+\gamma)} - \frac{1}{6} \varpi_{1,\ell}^{(1+\gamma)} \right] \\
&\quad + \frac{2+\gamma}{\ell+1} \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^{\ell+1} \varpi_{1,\ell}^{(1+\gamma)} \\
&= (2+\gamma) \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^\ell P(\ell, \gamma) \varpi_{1,\ell}^{(1+\gamma)} + \frac{2+\gamma}{\ell+1} \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^{\ell+1} \varpi_{1,\ell}^{(1+\gamma)} \\
&\quad + (2+\gamma) \left(\frac{3}{2}\right)^{1+\gamma} \left(\frac{1}{3}\right)^\ell \sum_{\ell_1=3}^{\ell-3} \frac{3^{\ell_1}}{\ell_1+1} \varpi_{1,\ell_1}^{(1+\gamma)} \varpi_{1,\ell-\ell_1}^{(1+\gamma)}.
\end{aligned}$$

Here

$$P(\ell, \gamma) = \left[ \frac{5}{6} - \frac{3\ell(\gamma+1)}{2(\ell-2-\gamma)} + \frac{3\gamma(\gamma+1)\ell(\ell-1)}{2(\ell-2-\gamma)(\ell-3-\gamma)} \right] + \frac{3^\ell}{\ell+1} \left[ 1 - \frac{(\ell+1)(\gamma+1)}{3(\ell-2-\gamma)} + \frac{\gamma(\gamma+1)\ell(\ell+1)}{18(\ell-2-\gamma)(\ell-3-\gamma)} \right].$$

Obviously, the last two terms in the right-hand side of the last equality are both nonnegative, so we only need prove that the factor  $P(\ell, \gamma)$  in the first term is nonnegative.

Let

$$P_1(\ell, \gamma) = \frac{5}{6} - \frac{3\ell(\gamma+1)}{2(\ell-2-\gamma)} + \frac{3\gamma(\gamma+1)\ell(\ell-1)}{2(\ell-2-\gamma)(\ell-3-\gamma)},$$

$$P_2(\ell, \gamma) = 1 - \frac{(\ell+1)(\gamma+1)}{3(\ell-2-\gamma)} + \frac{\gamma(\gamma+1)\ell(\ell+1)}{18(\ell-2-\gamma)(\ell-3-\gamma)},$$

then

$$P(\ell, \gamma) = P_1(\ell, \gamma) + \frac{3^\ell}{\ell+1} P_2(\ell, \gamma).$$

If  $\ell = 5$ , then

$$P_2(5, \gamma) = 1 - \frac{2(\gamma+1)}{(3-\gamma)} + \frac{5\gamma(\gamma+1)}{3(3-\gamma)(2-\gamma)} > 0.$$

Now we consider the case  $\ell \geq 6$ . Let

$$Q(x, \gamma) = 18(x-2-\gamma)(x-2.5-\gamma)P_3(x, \gamma), \quad x \in [6, \infty),$$

where

$$P_3(x, \gamma) = 1 - \frac{(x+1)(\gamma+1)}{3(x-2-\gamma)} + \frac{\gamma(\gamma+1)x(x+1)}{18(x-2-\gamma)(x-2.5-\gamma)}.$$

By simple calculations, one has

$$Q_{xx}(x, \gamma) = 2\gamma^2 - 10\gamma + 24 > 0, \quad 0 < \gamma < 1.$$

So  $Q_x(x, \gamma)$  is an increasing function and

$$Q_x(x, \gamma) \geq Q_x(6, \gamma) = 19\gamma^2 - 80\gamma + 72 > 0, \quad x \geq 6, \quad 0 < \gamma < 1.$$

It immediately follows that  $Q(x, \gamma)$  is an increasing function with respect to  $x$  and

$$Q(x, \gamma) \geq Q(6, \gamma) = 102\gamma^2 - 198\gamma + 105 > 0, \quad 0 < \gamma < 1,$$

i.e.,  $P_3(x, \gamma) > 0$ . Noticing  $P_2(\ell, \gamma) > P_3(\ell, \gamma)$  yields

$$P_2(\ell, \gamma) > 0, \quad \ell \in [5, \infty).$$

Therefore,

$$\begin{aligned} P(\ell, \gamma) &= P_1(\ell, \gamma) + \frac{3^\ell}{\ell+1} P_2(\ell, \gamma) \geq P_1(\ell, \gamma) + \frac{81}{2} P_2(\ell, \gamma) \\ &= \frac{124}{3} - \frac{3(\gamma+1)(10\ell+9)}{2(\ell-2-\gamma)} + \frac{3\ell\gamma(\gamma+1)(5\ell+1)}{4(\ell-2-\gamma)(\ell-3-\gamma)} \\ &> \frac{124}{3} - \frac{3(\gamma+1)(10\ell+9)}{2(\ell-2-\gamma)} + \frac{3\ell\gamma(\gamma+1)(5\ell+1)}{4(\ell-2-\gamma)^2}. \end{aligned}$$

When  $\ell = 5$ , we easily know that  $P(\ell, \gamma) > 0$  by direct calculations. Next, we discuss the case  $\ell \geq 6$ . Let

$$P_4(\ell, \gamma) = \frac{124}{3} - \frac{3(\gamma+1)(10\ell+9)}{2(\ell-2-\gamma)} + \frac{3\ell\gamma(\gamma+1)(5\ell+1)}{4(\ell-2-\gamma)^2},$$

and

$$R(x, \gamma) = 12(x-2-\gamma)^2 P_4(x, \gamma), \quad x \geq 6.$$

Differentiating twice with respect to  $x$  gives

$$R_{xx}(x, \gamma) = 90\gamma^2 - 270\gamma + 632,$$

which is positive when  $\gamma \in (0, 1)$ .

So,  $R_x(x, \gamma)$  is an increasing function and

$$R_x(x, \gamma) > R_x(6, \gamma) = 729\gamma^2 - 2225\gamma + 2006 > 0.$$

Furthermore,  $R(x, \gamma)$  is an increasing function as well and

$$R(x, \gamma) > R(6, \gamma) = 3412\gamma^2 - 6020\gamma + 2968 > 0, \quad x \in [6, \infty).$$

So,  $P_4(\ell, \gamma) > 0$  implies  $P(\ell, \gamma) > 0$ . It follows that  $\varpi_{2,\ell}^{(\alpha)} \geq \varpi_{2,\ell+1}^{(\alpha)}$  for  $\ell \geq 5$ . All this completes the proof.  $\square$

### Appendix B

Firstly, we list a lemma which comes from [12, 15].

*Lemma a.*

- (i)  $1 - x < \exp(-x)$  holds for  $0 < x < 1$ .
- (ii)  $1 - x > \exp(-2x)$  holds for  $0 < x \leq 0.7968$ .
- (iii)  $(1+x)^\alpha \geq 1 + \alpha x$  holds for  $\alpha \leq 0$  or  $\alpha \geq 1$ ,
- (iv)  $(1+x_1+x_2+x_3+\cdots+x_n) \leq (1+x_1)(1+x_2)(1+x_3)\cdots(1+x_s)$ , where  $x_m \geq -1$  and  $\text{sign}(x_m) = \text{sign}(x_n)$  for  $\forall 1 \leq m, n \leq s$ .

Proof of Theorem 2.5.

*Proof.* (1). In view of (ii) of the Lemma a, one has

$$\begin{aligned} |\varpi_{1,\ell}^{(\alpha)}| &= \left(1 - \frac{\alpha+1}{\ell}\right) |\varpi_{1,\ell-1}^{(\alpha)}| = \left(1 - \frac{\alpha+1}{\ell}\right) \left(1 - \frac{\alpha+1}{\ell-1}\right) \cdots \left(1 - \frac{\alpha+1}{3}\right) |\varpi_{1,2}^{(\alpha)}| \\ &\geq \exp\left(-\frac{2(\alpha+1)}{\ell}\right) \exp\left(-\frac{2(\alpha+1)}{\ell-1}\right) \cdots \exp\left(-\frac{2(\alpha+1)}{3}\right) |\varpi_{1,2}^{(\alpha)}| \\ &= \exp\left(-2(\alpha+1) \sum_{k=3}^{\ell} \frac{1}{k}\right) |\varpi_{1,2}^{(\alpha)}|. \end{aligned}$$

Note that

$$\sum_{k=3}^{\ell} \frac{1}{k} < \int_2^{\ell} \frac{1}{x} dx = \ln \frac{\ell}{2},$$

so we get

$$|\varpi_{1,\ell}^{(\alpha)}| > \exp\left(-2(\alpha+1) \ln \frac{\ell}{2}\right) |\varpi_{1,2}^{(\alpha)}| = \frac{\alpha(1-\alpha)}{2} \left(\frac{2}{\ell}\right)^{2(\alpha+1)}.$$

From [12], we know that

$$\left| \varpi_{1,\ell}^{(\alpha)} \right| < \frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}},$$

i.e.,

$$B_1^L(\alpha, \ell) < \left| \varpi_{1,\ell}^{(\alpha)} \right| < B_1^R(\alpha, \ell), \text{ where } B_1^L(\alpha, \ell) = \frac{\alpha(1-\alpha)}{2} \left( \frac{2}{\ell} \right)^{2(\alpha+1)},$$

$$B_1^R(\alpha, \ell) = \alpha \left( \frac{2}{\ell+1} \right)^{\alpha+1}, \quad \ell \geq 3.$$

(2). From (1), we have

$$\sum_{k=\ell}^{\infty} \left| \varpi_{1,k}^{(\alpha)} \right| > \alpha(1-\alpha) 2^{2\alpha+1} \sum_{k=\ell}^{\infty} \frac{1}{k^{2(\alpha+1)}}.$$

Because of

$$\sum_{k=\ell}^{\infty} \frac{1}{k^{2(\alpha+1)}} > \int_{\ell}^{\infty} \frac{1}{x^{2(\alpha+1)}} dx = \frac{1}{(2\alpha+1)\ell^{2\alpha+1}},$$

then we get

$$\sum_{k=\ell}^{\infty} \left| \varpi_{1,k}^{(\alpha)} \right| > \frac{\alpha(1-\alpha)}{2\alpha+1} \left( \frac{2}{\ell} \right)^{2\alpha+1}.$$

In addition, we also have [12]

$$\sum_{k=\ell}^{\infty} \left| \varpi_{1,k}^{(\alpha)} \right| < 2 \left( \frac{2}{\ell} \right)^{\alpha},$$

it follows that

$$S_1^L(\alpha, \ell) < \sum_{k=\ell}^{\infty} \left| \varpi_{1,k}^{(\alpha)} \right| < B_1^R(\alpha, \ell), \text{ where } S_1^L(\alpha, \ell) = \frac{\alpha(1-\alpha)}{2\alpha+1} \left( \frac{2}{\ell} \right)^{2\alpha+1},$$

$$B_1^R(\alpha, \ell) = 2 \left( \frac{2}{\ell} \right)^{\alpha}, \quad \ell \geq 3.$$

□

Proof of Theorem 2.6.

*Proof.* (1) Let

$$\alpha_{\ell} = \frac{12 \ln \frac{\ell}{2}}{2\pi^2 - 15} - 1.$$

Then from  $\alpha < \alpha_{\ell}$ , we can get

$$\left( \frac{\ell}{2} \right)^{1+\alpha} \exp \left( -(\alpha+1)^2 \left( \frac{\pi^2}{6} - \frac{5}{4} \right) \right) > 1,$$

i.e.,

$$\frac{B_1^L(\alpha, \ell)}{\bar{B}_1^L(\alpha, \ell)} < 1.$$

Obviously, the conditions for the above inequality are

$$\alpha_3 \approx 0.0267, \quad \alpha_4 \approx 0.7551, \quad \text{and } \alpha_5 \geq \frac{5}{2} \ln \frac{5}{2} - 1 > \frac{9}{4} - 1 > 1.$$

(2) Let

$$f(\alpha, \ell) = \frac{\tilde{B}_1^L(\alpha, \ell)}{B_1^L(\alpha, \ell)} = \frac{2\alpha + 1}{5\alpha} \left(\frac{\ell}{2}\right)^{1+\alpha},$$

and

$$g(\alpha, \ell) = \ln f(\alpha, \ell) = \ln(2\alpha + 1) - \ln(5\alpha) + (1 + \alpha) \ln \frac{\ell}{2}.$$

Then

$$g_\alpha(\alpha, 3) = \frac{2}{2\alpha + 1} - \frac{1}{\alpha} + \ln \frac{3}{2},$$

and

$$g_{\alpha\alpha}(\alpha, 3) = \frac{4\alpha + 1}{\alpha^2(2\alpha + 1)^2} > 0,$$

that is to say that the function  $g(\alpha, 3)$  has a minimum value and from the equation  $g_\alpha(\alpha, 3) = 0$  we know that the minimum point is

$$\alpha^* = \frac{-1 + \sqrt{1 + \frac{8}{\ln \frac{3}{2}}}}{4}.$$

Therefore,

$$f(\alpha, \ell) > f(\alpha, 3) \geq f(\alpha^*, 3) = \exp(g(\alpha^*, 3)) = \frac{2\alpha^* + 1}{5\alpha^*} \left(\frac{3}{2}\right)^{1+\alpha^*} \approx 1.3443 > 1,$$

i.e.,  $B_1^L(\alpha, \ell) < \tilde{B}_1^L(\alpha, \ell)$ . □

Proof of Theorem 2.7 is almost the same as that of Theorem 2.5.

Proof of Theorem 2.8.

*Proof.* (1) Note that  $\varpi_{2,\ell}^{(\alpha)} \leq 0$  for  $\ell \geq 3$  and  $\varpi_{1,\ell}^{(\alpha)} \leq 0$  for  $\ell \geq 1$  ( $0 < \alpha < 1$ ), then

$$\begin{aligned} |\varpi_{2,\ell}^{(\alpha)}| &= \left(\frac{3}{2}\right)^\alpha \left| \sum_{\ell_1=0}^{\ell} \left(\frac{1}{3}\right)^{\ell_1} \varpi_{1,\ell_1}^{(\alpha)} \varpi_{1,\ell-\ell_1}^{(\alpha)} \right| \\ &= \left(\frac{3}{2}\right)^\alpha \left| \left(1 + \left(\frac{1}{3}\right)^\ell\right) \varpi_{1,\ell}^{(\alpha)} + \sum_{\ell_1=1}^{\ell-1} \left(\frac{1}{3}\right)^{\ell_1} \varpi_{1,\ell_1}^{(\alpha)} \varpi_{1,\ell-\ell_1}^{(\alpha)} \right| \\ &= \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) |\varpi_{1,\ell}^{(\alpha)}| - \sum_{\ell_1=1}^{\ell-1} \left(\frac{1}{3}\right)^{\ell_1} |\varpi_{1,\ell_1}^{(\alpha)}| |\varpi_{1,\ell-\ell_1}^{(\alpha)}| \right]. \end{aligned}$$

From Lemma a, we know that

$$(1 + \ell_1)^{1+\alpha} (1 + (\ell - \ell_1))^{1+\alpha} \geq (1 + (\alpha + 1)\ell_1)(1 + (\alpha + 1)(\ell - \ell_1)) \geq 1 + (\alpha + 1)\ell$$

and

$$\ell_1^{2(1+\alpha)} (\ell - \ell_1)^{2(1+\alpha)} \leq \left( \left( \frac{\ell_1 + (\ell - \ell_1)}{2} \right)^2 \right)^{2(1+\alpha)} = \left( \frac{\ell}{2} \right)^{4(\alpha+1)}.$$

So,

$$\begin{aligned}
|\varpi_{2,\ell}^{(\alpha)}| &= \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) |\varpi_{1,\ell}^{(\alpha)}| - \sum_{\ell_1=1}^{\ell-1} \left(\frac{1}{3}\right)^{\ell_1} |\varpi_{1,\ell_1}^{(\alpha)}| |\varpi_{1,\ell-\ell_1}^{(\alpha)}| \right] \\
&\leq \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}} - \sum_{\ell_1=1}^{\ell-1} \left(\frac{1}{3}\right)^{\ell_1} \frac{\alpha^2 (1-\alpha)^2 4^{2\alpha+1}}{(\ell-\ell_1)^{2(\alpha+1)} \ell_1^{2(\alpha+1)}} \right] \\
&\leq \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}} - \frac{\alpha^2 (1-\alpha)^2 4^{2\alpha+1}}{2} \left(1 - \left(\frac{1}{3}\right)^{\ell-1}\right) \left(\frac{2}{\ell}\right)^{4(\alpha+1)} \right],
\end{aligned}$$

and

$$\begin{aligned}
|\varpi_{2,\ell}^{(\alpha)}| &= \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) |\varpi_{1,\ell}^{(\alpha)}| - \sum_{\ell_1=1}^{\ell-1} \left(\frac{1}{3}\right)^{\ell_1} |\varpi_{1,\ell_1}^{(\alpha)}| |\varpi_{1,\ell-\ell_1}^{(\alpha)}| \right] \\
&\geq \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha(1-\alpha)}{2} \left(\frac{2}{\ell}\right)^{2(1+\alpha)} \right. \\
&\quad \left. - \alpha^2 2^{2(\alpha+1)} \sum_{\ell_1=1}^{\ell-1} \left(\frac{1}{3}\right)^{\ell_1} \frac{1}{(1+\ell-\ell_1)^{\alpha+1} (1+\ell_1)^{\alpha+1}} \right] \\
&\geq \left(\frac{3}{2}\right)^\alpha \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha(1-\alpha)}{2} \left(\frac{2}{\ell}\right)^{2(1+\alpha)} - \left(1 - \left(\frac{1}{3}\right)^{\ell-1}\right) \frac{\alpha^2 2^{2\alpha+1}}{1 + (\alpha+1)\ell} \right].
\end{aligned}$$

(2) Note that  $\varpi_{2,\ell}^{(1+\alpha)} \geq 0$  for  $\ell \geq 4$  and  $\varpi_{1,\ell}^{(1+\alpha)} \geq 0$  for  $\ell \geq 2$  ( $0 < \alpha < 1$ ), then

$$\begin{aligned}
|\varpi_{2,\ell}^{(1+\alpha)}| &= \left(\frac{3}{2}\right)^{1+\alpha} \left| \sum_{\ell_1=0}^{\ell} \left(\frac{1}{3}\right)^{\ell_1} \varpi_{1,\ell_1}^{(1+\alpha)} \varpi_{1,\ell-\ell_1}^{(1+\alpha)} \right| \\
&= \left(\frac{3}{2}\right)^{1+\alpha} \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) |\varpi_{1,\ell}^{(1+\alpha)}| + \sum_{\ell_1=2}^{\ell-2} \left(\frac{1}{3}\right)^{\ell_1} |\varpi_{1,\ell_1}^{(1+\alpha)}| |\varpi_{1,\ell-\ell_1}^{(1+\alpha)}| \right. \\
&\quad \left. - \left(\frac{1}{3} + \left(\frac{1}{3}\right)^{\ell-1}\right) (1+\alpha) |\varpi_{1,\ell-1}^{(1+\alpha)}| \right].
\end{aligned}$$

So,

$$\begin{aligned}
|\varpi_{2,\ell}^{(1+\alpha)}| &\leq \left(\frac{3}{2}\right)^{1+\alpha} \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha(\alpha+1)3^{\alpha+2}}{2(\ell+1)^{\alpha+2}} + \sum_{\ell_1=2}^{\ell-2} \left(\frac{1}{3}\right)^{\ell_1} \frac{\alpha^2(1+\alpha)^2 3^{2(2+\alpha)}}{4(\ell_1+1)^{\alpha+2} (\ell-\ell_1+1)^{2+\alpha}} \right. \\
&\quad \left. - \frac{(1-\alpha)\alpha(1+\alpha)^2}{6} \left(\frac{3}{\ell-1}\right)^{2(2+\alpha)} \left(\frac{1}{3} + \left(\frac{1}{3}\right)^{\ell-1}\right) \right] \\
&\leq \left(\frac{3}{2}\right)^{1+\alpha} \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{\alpha(\alpha+1)3^{\alpha+2}}{2(\ell+1)^{\alpha+2}} + \left(1 - \left(\frac{1}{3}\right)^{\ell-3}\right) \frac{\alpha^2(1+\alpha)^2 3^{2(2+\alpha)}}{24(1+(2+\alpha)\ell)} \right. \\
&\quad \left. - \frac{(1-\alpha)\alpha(1+\alpha)^2}{6} \left(\frac{1}{3} + \left(\frac{1}{3}\right)^{\ell-1}\right) \left(\frac{3}{\ell-1}\right)^{2(2+\alpha)} \right],
\end{aligned}$$

and

$$\begin{aligned}
|\varpi_{2,\ell}^{(1+\alpha)}| \geq & \left(\frac{3}{2}\right)^{1+\alpha} \left[ \left(1 + \left(\frac{1}{3}\right)^\ell\right) \frac{(1-\alpha)\alpha(1+\alpha)}{6} \left(\frac{3}{\ell}\right)^{2(2+\alpha)} \right. \\
& + \frac{(1-\alpha)^2\alpha^2(1+\alpha)^2}{216} \left(1 - \left(\frac{1}{3}\right)^{\ell-3}\right) \left(\frac{6}{\ell}\right)^{4(2+\alpha)} \\
& \left. - \frac{\alpha(1+\alpha)^2}{2} \left(\frac{1}{3} + \left(\frac{1}{3}\right)^{\ell-1}\right) \left(\frac{3}{\ell}\right)^{2+\alpha} \right].
\end{aligned}$$

The proof is thus complete.  $\square$

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